ERIC Docum esume — See Instruction Sheet — DATE OF RESUME 7/1/65 — DATE RECEIVED 6/28/65

**A. PUBLICATION DATA**

1. ERIC NUMBER ED001718

2. O[illegible]INATOR CONTROL NO(S). N.A.

3. IDENTIFICATION NO. N.A.

4. AUTHOR(S): Board of Education of the City of New York

5. TITLE OF DOCUMENT ABSTRACTED: General Mathematics: 9th. Year

6. PUBLICATION TITLE: N.A.

7. EDITOR(S): N.A.

8. PLACE OF ORIGIN Board of Education of the City of New York
ORG.
ST. Brooklyn, New York
CITY, STATE
TELE. EXT.
INST/LOC CODE

9. PUBLISHER OR SOURCE Board of Education of the City of New York Brooklyn, New York

| | | | |
|---|---|---|---|
| 10. VOL. | 11. PAGES | 12. MO./YR. | 13. TOTAL PP. |
| N.A. | N.A. | 5/65 | 270 |

DO NOT WRITE IN SPACE BELOW

MICROFILM COPY AVAILABLE
YES ☒ NO ☐

COPYRIGHT GRANTED
YES ☒ NO ☐

COPYRIGHT RELEASE FILED
YES ☒ NO ☐

TYPE OF REPORT
PROPOSAL ☐
IN-PROGRESS ☐
FINAL ☒
FOLLOW-UP ☐

OTHER ERIC REFERENCES:

ERIC SATELLITE CODE

**B. INDEXING DATA**

14. SUBJECT AREA

15. TYPE OF ENDEAVOR curriculum guide

16. INDEXING TERMS (6-12)

| | | | |
|---|---|---|---|
| mathematics | problem solving | teacher guide | mathematical recreations |
| teacher evaluation | student self-evaluation | oral activities | secondary schools |

17. ABSTRACT (MAX. 250 WORDS)

A course is presented for students whose needs would not be served by regular 9th grade mathematics.

General mathematics as treated here is more than arithmetical computation. It contains elements of algebra, geometry, and trigonometry. The aim is to contribute to the general development of the student through: an appreciation and understanding of mathematical techniques and processes; an understanding of the application of mathematics to the individual's relationship to other fields of study; maintenance in each pupil of competence in number concepts and of accuracy in fundamental operations and problem solving; development of habits of accuracy, neatness, and orderliness; and development of an attitude of judgment based on facts.

Teaching suggestions are included. The exercises have been carefully graded to permit the teacher to select those best suited to the abilities and needs of the individual pupil. Methods of evaluation are also discussed. These include written activities such as tests, quizzes, and homework, and oral activities such as pupil justification of statements, pupil correction of each other's errors, and oral quizzes.

ERIC No.

18. SOURCE OF ABSTRACT

**C. FUNDING DATA**

19. SUPPORTING ORGANIZATION

20. DIVISION OR DEPT.

21. CONTRACT NO.

PROJECT NO.

22. TOTAL COST OF PROJECT

23. DURATION OR PROJECT

U.S. Department of Health, Education, and Welfare
Office of Education

ERIC Form 1

ED001718

CURRICULUM BULLETIN • 1964-65 SERIES • NO. 16

# GENERAL MATHEMATICS

# 9th Year

BOARD OF EDUCATION OF THE CITY OF NEW YORK

REPRINTED BY PERMISSION
U. S. DEPARTMENT OF HEALTH, EDUCATION AND WELFARE
OFFICE OF EDUCATION
THIS DOCUMENT HAS BEEN PRINTED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL OFFICE OF EDUCATION POSITION OR POLICY.

BOARD OF EDUCATION

James B. Donovan, President

Lloyd K. Garrison, Vice-President

Joseph G. Barkan
Aaron Brown
Thomas C. Burke
Alfred A. Giardino
Morris Iushewitz
Clarence Senior
Mrs. Rose Shapiro

---

Superintendent of Schools

Calvin E. Gross

Executive Deputy Superintendent of Schools

Bernard E. Donovan

Deputy Superintendents of Schools

Frederick W. Hill
John B. King
Joseph O. Loretan

Superintendent of Design, Construction and Physical Plant

Eugene E. Hult

---

DEPUTY SUPERINTENDENT'S CABINET

CURRICULUM DEVELOPMENT AND COORDINATION

Joseph O. Loretan, Chairman

Clare C. Baldwin
William H. Bristow
Sarah A. Cameron
John B. King
Richard M. Lubell
Carmella Nesi
Jacob H. Shack
Shelley Umans
Frederick H. Williams

Copyright 1965
By the Board of Education of the City of New York

Application for permission to reprint any section of this material should be made to the Superintendent of Schools, 110 Livingston Street, Brooklyn, N.Y. 11201. Reprint of any sections of this material shall carry the line, "Reprinted from (title of publication) by permission of the Board of Education of the City of New York."

Copies of this publication may be purchased from: Board of Education of the City of New York, Publications Sales Office, 110 Livingston Street, Brooklyn, N.Y. Checks should be made payable to: Auditor, Board of Education. Price: $2.00.

# FOREWORD

As a basic study, mathematics has much to contribute to meeting the demands and the opportunities of our changing society. Mathematics, with its structure and applications, is one of the cornerstones of a general education, which is the best insurance for helping individuals adapt to the changes of today and the needs of tomorrow.

General Mathematics: 9th Year is an alternate course for students whose needs would not be served by Mathematics: 9th Year as outlined in the Course of Study Mathematics Grades 7-8-9.

This publication is the result of continuing city-wide experimentation and continuous evaluation by teachers, supervisors, and mathematics coordinators.

We thank the staff members who have contributed so generously to this work.

JOSEPH O. LORETAN
Deputy Superintendent of Schools

January, 1965

# TABLE OF CONTENTS

Page

Introduction .................................................... 1
Aims and Objectives.............................................. 2
Suggestions for Teaching......................................... 3
Mathematical Recreations......................................... 4
A Plan for Using This Publication................................ 5
Evaluation....................................................... 6
Development of General Mathematics: 9th Year..................... 7

Chapter:

I. EQUATIONS........................................ 10
II. DIRECT MEASUREMENT.............................. 42
III. GEOMETRY UNIT................................... 58
IV. CONSUMER MATHEMATICS............................ 96
V. STATISTICS....................................... 124
VI. GRAPHS........................................... 166
VII. INDIRECT MEASUREMENT............................. 186
VIII. CONTEMPORARY MATHEMATICS - OTHER NUMBER BASES......... 227

## INTRODUCTION

Mathematics is a basic study which has much to contribute to a better understanding of life in a society which is changing so rapidly in scientific, industrial, and social areas. Our pupils need basic mathematical knowledge which will give them the flexibility they must have to adapt to a changing society. There are continuing elements in mathematics which the school must maintain, strengthen, and extend, while providing a general mathematics education which will help the pupils become mathematically literate citizens.

It has become evident over the years that no one mathematics course of study in ninth year could adequately serve the needs of all pupils in this grade. Therefore, different mathematics courses have been set up so that pupils would have an opportunity to benefit better from a particular course which is most appropriate for them in terms of their needs, interests, and abilities.

This bulletin is the product of several years of experimentation, first in selected schools, and then on a city-wide basis in ninth year General Mathematics classes of New York City. Previously, these materials were available in the form of eight mimeographed units. Through a continuous program of evaluation by teachers, supervisors, and mathematics coordinators, the present bulletin evolved. Care has been taken to provide adequate articulation with General Mathematics: 10th Year.

The general education program of which this syllabus is a part is intended to give the individual an understanding of the world in which he lives in its many quantitative aspects. This course as a part of the general education program is designed to develop those concepts and tools which concern the phases of living which require quantitative and spatial thinking. The processes of mathematics used in dealing with the quantitative and spatial aspects of life include more than just arithmetical computation. General mathematics as here defined contains elements of arithmetic, algebra, geometry and trigonometry which contribute to such understanding. Many pupils who have not been successful with mathematics should find in this course experiences which will challenge their interest.

## Aims and Objectives

A determination of objectives must consider the general needs and the special needs of these pupils. The first requires appreciation of the values and contributions of simple mathematics to mankind; the second requires the development of skills which may be used in the personal and social lives of pupils.

The aim of the course is to contribute to the general development of the student through:

1. An appreciation and understanding of mathematical techniques and processes.
2. Understanding of the application of mathematics in the individual's relationships to government, industry, science, insurance, banking, personal accounting, and in the use of instruments of measurement, graphs, tables, and charts.
3. Maintenance in each pupil of competence in number concepts, accuracy and speed in fundamental operations, and in problem solution. (Arithmetic is stressed in each unit.)
4. Development of habits of accuracy, neatness, orderliness, and systematic procedures, and an attitude of judgment based on facts.

A major objective of this course is the improvement of the pupil's ability to understand and use those arithmetic skills which he will need to take his place as a self-sustaining citizen in the world of today and tomorrow. To achieve this purpose, throughout the bulletin adequate exercises in the use of fundamental skills have been provided, usually in real life situations based upon the students' level of maturity. In addition, adequate consideration has been given to the important aspects on non-computational mathematics. Since these play a major role in the everyday life of the average citizen, they form a necessary background of mathematical knowledge for all students. Some of these are: the appreciation of the approximate nature of measurement, the recognition and appreciation of geometric forms in his environment, and the ability to read and interpret graphs and charts.

## Suggestions for Teaching

Pupils who are unable to master mathematics courses are often frightened and inhibited by the usual mathematical problems or exercises. It is, therefore, often necessary for the teacher to rebuild the confidence and self-respect of his pupils. He may need to imbue them with a more favorable attitude toward school and to show them the worthwhileness and the practical value of secondary school education. To achieve this, it is essential to know the characteristics of the pupils in the class, and to plan a program of instruction in mathematics that will capture their interest.

Students enrolled in general mathematics courses will, as a rule, need more help in forming associations between words and ideas in mathematics. All will learn more readily through concrete, practical presentation. A large number of concrete examples may be necessary to help them generalize correctly. It will be found that some of these pupils will not be able to concentrate as long or as intensely on a single topic in mathematics as other pupils. This course offers the teacher a variety of concrete illustrations of each principle that is being taught in order to build and hold the pupils' interest. For some pupils frequent changes of activities are essential. Therefore, provision is made for individual practice, group discussion, laboratory activity, some manual activity, more supervised study, individual projects, group projects, films, board work, and so on.

The exercises in the bulletin have been carefully graded to permit the teacher to select those best suited to the abilities and needs of the individual pupil. Provision has been made to challenge the more capable student by the inclusion of optional topics.

In suggesting procedures for teaching, a conscious effort has been made to illustrate the level of language suited to the understanding of the pupil. At the same time, precise mathematical terminology is employed to help the pupil improve his ability to read mathematical materials with understanding. It is expected that the teacher will provide for these vocabulary needs in her daily plans.

The applications suggested in this course will be found to be more direct than in most courses in mathematics. Stimulating motivations, applications, and visual aids are more generally used to give a direct appreciation of the role of mathematics in understanding the environment. There is emphasis on the concrete and practical. A large number of ordinary practical uses of mathematics is suggested. In all that is done, it is important that pupils acquire a sense of accomplishment and satisfaction. In the application of this course to an individual classroom, the tempo should be such that pupils will not feel rushed. They should have an opportunity to understand and work with the principles that are presented. They should be given a feeling of success and an understanding of what they have studied. Teachers may use a spiral approach and return to work of earlier units throughout the year.

Facility in the use of fundamental processes in arithmetic is an integral part of this course and permeates almost every phase of it. For this reason it is not indicated as a special topic. However, it should be recognized that this course embraces a planned program for the maintenance or reteaching of arithmetical skills and concepts.

Understanding of basic concepts must precede intensive drill. This often will reduce the amount of drill needed to fix a skill and also will reduce the rate of forgetting. It should be recognized that the rationalizing process has two vital elements: the concrete and the general. Many varied concrete instances are needed to establish a generalization. True understanding can best be achieved when the pupil himself is encouraged to make the generalization as based on concrete instances.

The teacher should keep in mind that mathematics is more than a number system and operations. It is also a method of thinking and, as such, gives basic training for sound thinking. Overemphasis on the computational aspect of arithmetic may produce a meaningless skill. It should be related to personal and social needs and to social and informational values. The tendency to think quantitatively should be encouraged.

The decision to include the chapter on Contemporary Mathematics is based on the experiences of many teachers. This experience has shown that the pupils find the study of base five and base two numeration systems stimulating and well within their ability to comprehend. A more important consideration, however, is that through the study of these "strange" numeration systems, new light is shed on the structure of the decimal system. A deeper understanding of the importance of place value and the significance and need for regrouping and exchange in the operations of arithmetic is a natural outgrowth of these experiences.

## Mathematical Recreations

Although they are not mentioned in this course as a special topic, it is recommended that mathematical recreations be presented at regular intervals. It has been found that these recreations have special values for the student in the general course.

1. They stimulate his interest in numbers. This interest may have been dulled by many failures. They may even arouse his curiosities about number relations.

2. The mathematical recreation is a good setting for creating a social situation. Questions, answers, class discussion and relating of similar instances are natural activities in the presentation and discussion of mathematical recreations. Pupils should be encouraged to participate in exercises in mathematical self-expression.

## A Plan For Using This Publication

It is suggested that teachers and supervisors consider the following in using this publication:

Review the entire bulletin before making plans to teach. Read each chapter in turn completely to become acquainted with the content and flavor of General Mathematics: 9th Year, and with the relationship between the topics in the course.

Study the chapter containing the topics you plan to present. Study the suggested procedure for the development of each topic. Make a tentative division of the topic into class lessons. Use the suggested procedures as an aid in preparing lesson plans.

Amplify the practice material suggested for each topic with additional material from various textbooks.

## Evaluation

An evaluation program includes not only the checking of completed work at convenient intervals, but also continual appraisal. It is a general principle of evaluation that results are checked against objectives. The objectives of this course include concepts, principles, and understandings, as well as basic skills.

Written tests are the most used instrument for evaluation and remain the chief rating tools of the teacher. Test items should be designed to test not only recall of factual items, but also the ability of the pupil to make intelligent use of facts. Some of the written activities which teachers may use for the purpose of evaluation include:

written tests
written homework assignments
notebooks
board work
special reports
quizzes

To continually evaluate pupil understanding, there are a number of oral activities which teachers may use including the following:

pupil explanations of approaches used in new situations
pupil justification of statements
pupil restatement of problems
pupil statements of interrelationship of ideas
pupil discovery of patterns
oral quizzes
reports
pupil correction of each other's errors

Evaluation procedures also include teacher's observation of pupil's work at chalkboard and pupil's work at seat.

Self-evaluation by pupils can be encouraged through short self-marking quizzes.

## Development of General Mathematics: 9th Year

The Curriculum Council designated an Ad Hoc Committee to review the program of General Mathematics: 9th Year.

The Committee met and recommended that a smaller committee be formed to review the existing course of study in General Mathematics for the 9th Year and to prepare a manual of teaching procedures.

The Curriculum Council approved the proposal that a committee be formed to plan the general design for a program of research, evaluation, articulation, and production in connection with General Mathematics for the 9th Year.

The project was initiated by Joseph O. Loretan while he was in charge of the Junior High School Division. It was carried out under the general direction of Martha R. Finkler, Acting Associate Superintendent, Junior High School Division; Margaret Bible, Assistant Superintendent, Junior High School Division; Maurice Hopkins, Acting Associate Superintendent, High School Division; Seelig Lester, Assistant Superintendent, High School Division; Jacob H. Shack, Acting Associate Superintendent, Division of Curriculum Development; and William H. Bristow, Assistant Superintendent, Bureau of Curriculum Research.

An advisory committee was set up which consisted of:

Louis Braverman, Retired; Max Bromer, Chairman of Related Technology, William E. Grady High School; Laura Eads, Research Associate, Bureau of Curriculum Research; Morris Hertzig, Principal, Robert A. Van Wyck Junior High School; Ella Huerstel, Supervisor of Elementary Mathematics, Elementary School Division; Saul Landau, Chairman of Mathematics, James Monroe High School; Anne Peskin, Assistant Principal (on leave); Max Peters, Chairman of Mathematics, (Chairman of Committee), George W. Wingate High School; Leonard Simon, Curriculum Coordinator for Junior High Schools, Bureau of Curriculum Research; Frank Wohlfort, Supervisor of Mathematics Coordinators, Junior High School Division.

Junior High School Mathematics Coordinators wrote experimental units for classroom tryout. The coordinators involved in writing materials are:

J. Spencer Abbott, Florence Apperman, Samuel Dreskin, Helen Halliday, Miriam S. Newman, Alfred Okin, Meyer Rosenspan, Benedict A. Rubino, Murray Soffer, Bertha Weiss.

Former Junior High School Mathematics Coordinators who assisted in writing unit materials were:

Ruth W. Gallagher, Marion H. Kiernan, Mary Q. Mortola (deceased), Anne Peskin.

Valuable suggestions were received from these Junior High School Mathematics Coordinators:

Henrietta Antoville, Samuel Bier, Joseph Gehringer, Charles J. Goode, Helen Kaufman, Rose Klein, George Paley, Joseph Segal, Ada Sheridan.

As the experimental units were written, a review committee of Mr. Bromer, Mr. Peters, Mr. Simon and Mr. Wohlfort reviewed the units and made suggestions.

The final materials were written during the summer of 1964 by Junior High School Mathematics Coordinators: Helen Halliday and Meyer Rosenspan. Mr. Wohlfort was project director for the summer work.

George Grossman, Acting Director of Mathematics, reviewed the final materials.

As each experimental unit was completed, it was subjected to classroom tryout. The initial tryout was restricted to 23 junior high schools for evaluation. The teachers who took part in this classroom tryout are:

Mrs. Birkhaur, Mr. Boyarsky, Mr. de Mattos, Miss Eskowitz, Mr. Gallagher, Mr. Goodfriend, Mrs. Holland, Mr. Kennedy, Mr. Ledee, Mr. Levy, Mr. Mara, Mr. Maurer, Mrs. Meyerson, Mr. Molin, Mr. Reinfelder, Mr. Rohatyn, Mr. Saiken, Mr. Sanders, Mr. Siegel, Mrs. Weintraub, Miss Wohlberg.

Others who contributed suggestions are the following:

Abell, C.
Addleston, L.
Adelsohn, C.
Arvamitis, J.C.
Attanasio, S.
Bailey, H.
Banner, M.
Barber, W.
Barkin, A.
Barr, R.
Barrett, R.
Barton, S.
Baxter, J.
Becker, F.S.
Beckler, A.
Berkhout, P.G.
Berkley, A.
Berman, D.
Bernstein, E.
Blitz, S.
Bold, B.
Bowdkin, J.
Brown, R.
Bruno, J.
Cahill, J.R.
Campora, G.
Carson, D.T.
Cohen, L.
Conte, A.
Contrada, J.
Cutropia, A.
David, R.
DeLise, M.
Dodes, I.
Dorkin, P.
D'Orlando, A.
Dowd, M.
Edelman, S.J.
Edelson, M.
Epstein, D.T.
Ettinger, M.H.
Fabbricante, O.E.
Feinstein, S.
Feldman, M.
Fine, R.L.
Fisch, D.
Flaumenhaft, P.
Fogel, S.
Fox, L.
Friedman, F.
Friedman, P.
Frome, E.
Funke, B.
Gastwirth, P.
Gaswirth, I.
Gibbs, P.
Goldenberg, N.
Goldstein, D.
Goodfriend, L.
Goodman, S.J.
Green, S.
Grossman, G.
Gruber, J.
Grumet, M.
Haggerty, G.G.
Haimowitz, I.C.
Hall, D.
Halley, V.L.
Hanauer, M.
Handrakis, P.
Harrigan, E.M.
Harris, R.N.
Hees, L.
Herselle, D.
Hertzig, M.
Hlavaty, J.
Hogan, J.
Holland, R.A.
Huggard, E.M.
Hughson, A.
Hurwith, G.L.
Jackson, D.
Jacobs, B.
Jacobs, S.J.

Jacobs, S.
Jacobson, M.
Jacobson, R.
Jacoby, R.V.
Joffe, L.M.
Kapp, M.
Kaslow, F.
Kaufman, C.B.
Kennedy, M.
Kennelly, D.
Kirwin, M.C.
Klein, I.
Korman, M.
Krausner, J.
Kravitz, R.
Kravitz, J.
Landau, S.
Lande, L.S.
Leahy, Jr., G.H.
Leder, R.
Leventon, A.
Levine, M.M.
Levy, J.
Libby, E.R.
Liebling, L.
LoFrumento, A.
London, B.
Mafson, S.
Maisels, M.
Manfesi, G.C.
Mettal, W.
Meyer, S.
Meyerson, F.
Mondiva, M.
Milhaven, E.B.
Mintz, E.
Mintz, P.
Moncynik, H.
Morris, C.
Moskowitz, A.
Murray, J.E.
Nadler, M.
Nelson, E.C.
Nitzschke, F.W.
Olsen, M.G.
Papernik, I.
Pedowitz, M.
Peiser, L.M.
Petcher, E.M.
Pozzioli, E.L.
Rabowski, F.
Reinfelder, A.
Reiss, N.
Rhodes, B.
Rind, W.
Rosato, I.
Rose, W.
Rosen, H.S.
Rosenbaum, R.
Rosofsky, J.W.
Roter, S.
Rubin, J.
Sacco, M.
Saiken, M.
Saltzman, S.D.
Schacter, S.
Schaier, F.B.
Schiff, H.
Schoenberg, F.
Schoenholtz, H.
Scholder, H.
Schwartz, A.
Schwartz, D.
Schwartz, M.
Shakin, M.
Sharp, Lee H.
Shuger, A.
Siegel, S.
Silverman, M.
Smith, C.
Smyth, V.
Somerdin, J.
Stack, C.
Stein, E.
Stein, M.
Susman, S.H.
Swirin, M.
Tepper, M.A.
Testa, C.V.
Troy, Z.
Wantman, R.M.
Wayne, A.
Wecker, M.S.
Weintraub, L.
Weiss, M.W.
Williams, A.L.
Williams, M.
Wolin, J.
Yacavone, M.
Zuckerberg, J.
Zweiter, M.

Grateful acknowledgment is expressed to the many who have contributed to this bulletin: the present and past members of the Junior High School Mathematics Curriculum Committee, and the High School Standing Committee on Mathematics, as well as principals of experimental schools who arranged schedules.

A special acknowledgment is made to the following: Benedict A. Rubino and Alfred Okin who prepared the art work; Simon Shulman who designed the cover; and Maurice Basseches, Editor, who had overall responsibility for design and production.

# I.

# EQUATIONS

## I. Examining Sentences

### A. Recognizing sentences and non-sentences

1. Non-mathematical

   a. Experience

      1) Have a pupil walk to the door and open it.
      2) Have other pupils tell what he has done and list replies on the board:

         Examples:

         He opened the door.
         He walked to the door and opened it.
         He opened it.
         He walked slowly.

      3) Elicit from pupils that these are all sentences because each has a subject and a predicate.

         Note: Some may not be sentences. This is discussed in b.

   b. Have pupils discuss their ideas about rockets. "What can you tell the class about rockets?"

      1) Possible answers:

         The government is experimenting with rockets.
         Long range rockets
         Expensive
         A toy balloon is a kind of rocket.

      2) Have pupil select the sentences.
      3) Have him recognize that "long range rockets" and "expensive" are not sentences. (A sentence expresses a complete thought.)

2. Mathematical

   a. Have pupil decide which of the following are sentences:

      $8 + 7 = 15$
      $6 + 2 = 3 + 5$
      $5 \times 7$
      $\frac{18}{9}$
      $18 = 20 - 2$
      a right angle
      the mode
      $\frac{10 + 20}{2} = 15$

b. Have pupil give illustrations of

1) mathematical sentences
2) mathematical expressions which are not sentences.

B. Judging the truth or falsity of statements

Note: In mathematics, a declarative sentence which can be judged true or false is called a statement.

1. General statements

Tell pupil that he knows that statements are sometimes made which are not true.

a. Have pupil judge truth or falsity of the following statements:

1) Mickey Mantle plays for the Giants.
2) In the United States, Columbus Day is celebrated on October 12.
3) The capital of New York State is New York City.
4) The moon is made of green cheese.
5) A Cadillac is an expensive car.
6) A magnet attracts aluminum.

b. Have pupil make up statements and tell whether they are true or false.

2. Mathematical statements

a. Judge the truth or falsity of the following statements:

1) A yard equals three feet.
2) A dime equals ten cents.
3) A foot equals fifteen inches.
4) A dollar equals fifty cents.
5) $3 + 5 = 8$
6) $\frac{15}{3} = 4$
7) $3 \times 6 = 9 \times 2$
8) $\frac{1}{2} + \frac{1}{2} = \frac{5}{8} + \frac{2}{8}$
9) $\frac{1}{2} = 75\%$
10) $.5 + .3 = .8$

b. Have pupil realize that merely saying "equal" or using the symbol "=" does not mean that the statement is true.

c. Suggested Practice

1) Label each statement true or false.

$4 + 4 = 3 + 5$

$\frac{68}{2} = 34$

$10 \times 25 = 2500$

$39 + 46 = 83 + 1$

$100 - 36 = 60 + 4$

$1.0 - .1 = .8 + .1$

$\frac{1.0}{2} = .5$

$1.3 + .4 = 1.8 - .2$

2) Have pupil make up illustrations of true and false mathematical statements.

C. Open sentences

Completion sentences are examples of open sentences. They are neither true nor false. When the blank is filled in, the sentence becomes a statement which is true or false depending on the replacement used.

Question: Can we always decide whether a sentence is true or false?

1. Have pupil fill in the blanks:

   a. A pupil in our class is _____.
   b. _____ is an officer of the class.
   c. _____ is the G.O. representative.

   Guide pupils to compare the number of choices he has for answers to b and c. Have him discover that whereas many answers are correct for a, there are just two or three answers to b, and only one for c.

2. Practice

   a. Have pupil make a list of all possible answers that will make the following statements true:

      1) _____ are the teachers of our class.
      2) _____ is the mathematics teacher of our class.

   b. Have pupils practice formulating open sentences in which

      1) there are several possible answers
      2) there is just one possible answer.

D. Open sentences as related to mathematics

1. Have pupil replace the frame with the numerals which will make each sentence a true statement.

| Column A | Column B |
|---|---|
| $\square + 16 = 18$ | $\square + \square = 13$ |
| $16 - \square = 10$ | $26 - \square = \square$ |
| $8 \times 7 = \square$ | $\square \times 7 = \bigcirc$ |
| $\frac{72}{\square} = 9$ | $\frac{\square}{\triangle} = 4$ |
| $1.5 + 3.7 = \square$ | $\frac{\triangle}{\square} = .2$ |

**Guide pupils to compare the number of choices he has for answers to Column A and the number of choices he has for Column B. Have him discover that whereas there are several pairs of numbers which will make the B statements true, there is only one number which will make the A statements true in each case.**

2. Have pupil try formulating open sentences in mathematics in which

   a. there are several possible answers $16 = \square + \square$

   b. there is just one possible answer. $16 = 4 + \square$

II. Use of a Letter to Represent a "Number to be Found"

A. Letters and numbers used in codes

A furniture dealer marks his price labels in code. His code word is SCUBA DIVER. The letters and their corresponding numbers are shown below.

S C U B A D I V E R

0 1 2 3 4 5 6 7 8 9

1. Using the code, find the cost of (the last two letters in the coded price represent cents):

   a. a set of dining room furniture marked CRRRD
   b. a sofa marked CVDSS
   c. a chair marked BRDS
   d. a lamp marked CRSS

2. What letters would be used to represent items valued at

   a. \$48.95
   b. \$134.99
   c. \$1939.89
   d. \$267.49

3. Have pupil make up his own code and use the letters in the code to represent numbers. (He may use ten different letters in his name, and pair them with numbers 0-9.)

B. Letter symbols used to replace blanks in mathematical sentences

1. Review with pupil that letters can be used in place of numbers. (See II-A)

2. Extend development.

   a. _____ is the total number of 9th grade pupils in the school.

   Guide pupils to see that it is helpful to use a letter in place of a number if we do not know at first what the number is. The total number of 9th grade pupils in this school can be represented by the letter n.

b. _____ is the number of inches (to the nearest inch) in the width of my desk. "b" represents the number of units (to the nearest inch) in the width of my desk.

c. Emphasize with the pupils that the letter represents a number. When we use expressions such as length, age, height, weight, money, we mean the number of inches in the length, the number of years, the number of inches in height, and so on.

Thus: $n$ = number of inches in the width of the desk
$n$ = number of dollars in the amount of money
$n$ = number of years

However, in any given problem, different letters represent numbers of different things.

d. Guide pupil to see:

1) Blanks in mathematical sentences can be replaced by a letter symbol until the number is discovered. Here n stands for the number until we find it.

2) Mathematical sentences containing letters or other symbols in place of numerals are open sentences.

a) $3 + \square = 12$

$3 + n = 12$

b. $\square + \triangle = 16$

$n + x = 16$ (Use different letters because the numbers may be different.)

3) Develop skill in algebraic representation through many exercises such as the following:

If n represents a number, express in terms of n

a. 3 times the number
b. $\frac{3}{4}$ of the number
c. the number increased by 6
d. the number decreased by 4
e) the number divided by 4
f) 6 subtracted from the number
g) 5 more than the number
h) the number diminished by 5
i) 5 less than the number

## . Equations

### A. Meaning of an equation

1. Statements of equality

   a. Have pupil consider the following statements:

   $3 + 5 = 8$
   $3 \times 6 = 9 \times 2$
   $4 + 5 = 9$
   $18 - 2 = 16$
   $\frac{140}{7} = 20$

   b. Elicit from pupil what is true of all these statements. (Each states that the two quantities are equal.)

   c. Have pupil consider each statement and conclude that each is a statement of equality.

   d. Tell pupil that in mathematics a statement of equality is called an equation.

2. Practice

   a. Which of these are equations?

   $8 + 5 = 13$
   $4 + 3$
   $74 - 10$
   $4 + 8 = 12$

   b. Have pupil give examples of equations.

### B. Members of an equation

1. Identifying members

   a. Have pupil note that a statement of equality may be considered to consist of two quantities, e.g., in the equation $6 + 2 = 8$, $6 + 2$ is one quantity, and 8 is the other.

   b. Have him read off the quantity on the left side of the equal sign; on the right side.

   c. Tell him that the quantity $6 + 2$ is called the left member of the equation. What is a good way of referring to the quantity on the right side?

2. Practice

   In each instance identify the left member and the right member of the equation.

   $3 + 5 = 8$
   $4 + 4 = 3 + 5$
   $16 = 8 \times 2$
   $4 \times 4 = 2 \times 8$
   $n = 4 + 2$
   $16 = 8x$
   $\frac{n}{2} = 5$
   $6 - 4 = n$

C. The meaning of root of an equation

1. Have pupil find the value of the letter which will make the equation a true statement (by trial and error) using the set of numbers 1 through 20.

   a. In $n + 8 = 10$, 2 is the value of n which makes the equation true.

   b. In the equation $b - 3 = 4$, b has the value of 7 because $7 - 3 = 4$.

2. Have pupil arrive at the understanding that the value of the letter which makes the equation a true statement is called the root of the equation.

   a. In $n + 8 = 10$; when n is replaced by 2, $2 + 8 = 10$ is a true statement. 2 is called the root of the equation.

   b. In $b - 3 = 4$; when b is replaced by 7, $7 - 3 = 4$ is a true statement. 7 is the root of the equation.

3. Practice

   Have pupil study the following equations and find by trial and error the root of each:

   a. $q - 9 = 3$
   b. $6 + x = 30$
   c. $3y = 21$
   d. $\frac{x}{2} = 4$
   e. $y + \frac{1}{4} = \frac{1}{2}$
   f. $2.5 + n = 3$
   g. $7x = 35$
   h. $4 = \frac{m}{5}$

## IV. Solving Equations by Addition or Subtraction

A. Finding the roots of equations

1. Equivalent equations

   a. Have pupil find the root in each of the following using the set of numbers 1 through 20.

$$\begin{aligned} x + 7 &= 12 \\ x + 6 &= 11 \\ x + 3 &= 8 \\ x + 1 &= 6 \\ x &= 5 \end{aligned}$$

   b. Have pupil observe that in each equation the root is 5.

   c. Tell pupil that equations in the same **letter** which have the same root are called equivalent equations.

   d. Have pupil select from this group of equations the equation in which the root is obvious ($x = 5$).

e. Tell the pupil that to "solve" an equation means to change the given equation where the solution is not obvious to an equivalent equation where the solution is obvious, for example, $x = 2$, $x = 25$, $m = 14$.

2. Patterns for finding the root of an equation by subtraction

   a. Have pupil consider the following equations:

      1) Illustrations

| | |
|---|---|
| $5 + 7 = 12$ | True statement |
| $5 + 7 - 7 = 12 - 7$ | |
| $5 = 5$ | True statement |
| $8 + 6 = 14$ | True statement |
| $8 + 6 - 6 = 14 - 6$ | |
| $8 = 8$ | True statement |
| $10 + 8 = 18$ | True statement |
| $10 + 8 - 8 = 18 - 8$ | |
| $10 = 10$ | True statement |

      2) Have pupil see that if we start with a true statement and subtract

         a) the same number from each member, we arrive at a true statement
         b) subtract one of the addends from each member, we get the other addend (subtraction law).

   b. Have pupil consider equations containing **a letter.**

      1) Equations - IV-A-1-a

| | |
|---|---|
| $x + 7 = 12$ | Open sentence |
| $x + 7 - 7 = 12 - 7$ | Subtraction law (subtract 7 from each member) |
| $x = 5$ | Equivalent equation in which root is obvious |

**Check:**

$x + 7 = 12$
$5 + 7 = 12$
$12 = 12$

Have pupil note that 5 is the root of the equation in $x + 7 = 12$. To check whether the root is 5, we replace **the letter** with 5. $5 + 7 = 12$ is a true statement.

      2) $x + 6 = 14$
         $x + 6 - 6 = 14 - 6$
         $x = 8$

Have pupil note that 8 is a root of the equation $x + 6 = 14$. Check.

3) $x + 8 = 18$
$x + 8 - 8 = 18 - 8$
$x = 10$

**Have pupil note that 10 is a root of the equation $x + 8 = 18$. Check.**

4) $x + 1 = 6$
$x + 1 - 1 = 6 - 1$
$x = 5$

**Have pupil note that 5 is a root of the equation $x + 1 = 6$. Check.**

c. **Have pupil conclude and state in his own words that if the same number is subtracted from both members of an equation, the result is an equivalent equation.**

3. In a similar manner as in IV-A-2, develop with pupil the principle of solution of equations by addition.

a. Illustrations

| | |
|---|---|
| $7 - 5 = 2$ | True statement |
| $7 - 5 + 5 = 2 + 5$ | |
| $7 = 7$ | True statement |
| $x - 5 = 2$ | Equation |
| $x - 5 + 5 = 2 + 5$ | |
| $x = 7$ | Equivalent equation |

Check:
$x - 5 = 2$
$7 - 5 \overset{?}{=} 2$
$2 = 2$

7 is the root of the equation $x - 5 = 2$.

b. **After a number of such illustrations, have pupil conclude and state in his own words that if we add the same number to both members of an equation, the result is an equivalent equation.**

B. Practice

Find the root of each of these equations and check:

$x - 18 = 49$
$x + 17 = 35$
$n - 2\frac{1}{2} = 5\frac{1}{2}$
$n + 1\frac{1}{4} = 2\frac{1}{2}$
$y - .3 = 1.8$
$k - 1.1 = 3.5$
$a + 18.7 = 19.3$
$c + 3.75 = 8.50$
$y + 6.15 = 13.18$
$x - 9.01 = 25.$
$y + 20 \frac{3}{4} = 50$
$p + 30 \frac{5}{8} = 40$
$p - 16.85 = 13$

## V. Solving Equations by Multiplication or Division

### A. Finding the roots of equations

1. Equivalent equations

   a. Have pupil find the root in each of the following:

      $12x = 36$
      $6x = 18$
      $2x = 6$
      $x = 3$

   b. Have pupil observe that in these equivalent equations the root is 3.

   c. Have pupil select from this group of equations the equivalent equations in which the root is obvious $(x = 3)$.

2. Meaning of a coefficient

   a. Have pupil write 3 times a number in algebraic symbolism, e.g., 3n. Tell pupil that in the **product** 3n, 3 is called the coefficient of n.

   b. Have pupil select the coefficient of the **letter** in each of the following expressions:

| | | | |
|---|---|---|---|
| $5a$ | $2y$ | $\frac{1}{2}c$ | $1.5x$ |
| $\frac{3}{4}f$ | $2.1\ b$ | $18y$ | $1\frac{1}{2}n$ |

3. Patterns

   a. Review the meaning of factor. (Grade 7 bulletin, p. 40)

   b. Have pupil consider the following equations:

      1) Illustrations

$12 \times 3 = 36$ True statement

$$\frac{12 \times 3}{12} = \frac{36}{12}$$

$3 = 3$ True statement

$6 \times 5 = 30$ True statement

$$\frac{6 \times 5}{6} = \frac{30}{6}$$

$5 = 5$ True statement

$2 \times 8 = 16$ True statement

$\frac{2 \times 8}{2} = \frac{16}{2}$

$8 = 8$ True statement

2) Have pupil see that if we start with a true statement and

   a) divide each member by the same number, we arrive at a true statement
   b) divide by one of the factors, we get the other factor (Division law).

c. Have pupil solve equations containing one **letter.**

1) Equations V-A-1-a

| | |
|---|---|
| $12x = 36$ | Equation |
| $\frac{12x}{12} = \frac{36}{12}$ | Division law (divide by 12) |
| $x = 3$ | Equivalent equation in which the root is obvious. |

Answer: the root is 3.

Check:

$$\begin{aligned} 12x &= 36 \\ 12(3) &\overset{?}{=} 36 \\ 36 &= 36 \end{aligned}$$

Have pupil note that 3 is the root of the equation $12x = 36$.
Have pupil check 3 as the root of the equation $12x = 36$.

$6x = 30$

$\frac{6x}{6} = \frac{30}{6}$

$x = 5$

Answer: the root is 5.

Check: as in c-1) above

Have pupil note that 3 is the root of the equation $6x = 18$.

$2x = 16$

$\frac{2x}{2} = \frac{16}{2}$

$x = 8$

Check: as in c-1) above

2) **Have pupil conclude and state in his own words that if both members of an equation are divided by the same number, the result is an equivalent equation.**

d. Practice. Find the root of each equation and check.

1) $3x = 6$ 2) $14x = 56$ 3) $3d = 2$ 4) $5x = 8$

5) $5x = 12$ 6) $2y = 6.4$ 7) $3x = 1.41$ 8) $.2b = 64$

9) $.03x = .21$ 10) $.5x = 12$ 11) $.6x = 1.8$ 12) $.15x = 3$

4. In a similar manner, develop with pupil the principle of solution of equations by multiplication.

a. Illustration

Have class compare these two statements.

$$\frac{12}{3} = 4$$

$$3 \times \frac{12}{3} = 3 \times 4$$

Are they both true? (Yes, 12 = 12.)

Develop with the pupils the possibility of solving an open sentence by using multiplication.

| | |
|---|---|
| $\frac{n}{3} = 4$ | **Equation** |
| $3 \times \frac{n}{3} = 3 \times 4$ | Equivalent equation |
| $n = 12$ | Equivalent equation in which the solution is obvious. |
| 12 | Is the root of the equation $\frac{n}{3} = 4$? |

Check:

$$\frac{n}{3} = 4$$

$$\frac{12}{3} \overset{?}{=} 4$$

$$4 = 4$$

b. **After a number of such illustrations, have pupil conclude and state in his own words that if both members of an equation are multiplied by the same number, (not zero), the result is an equivalent equation.**

B. Practice

1. $\frac{x}{2} = 4$  2. $\frac{n}{4} = 3\frac{1}{2}$  3. $\frac{1}{4}g = 6\frac{1}{2}$  4. $\frac{c}{3} = 4.6$  5. $\frac{1}{5}k = 1.5$
6. $\frac{x}{7} = 8.9$  7. $\frac{x}{2.5} = 7.4$  8. $2\frac{2}{5} = \frac{n}{15}$  9. $\frac{n}{.3} = 8.7$  10. $2.9 = \frac{y}{18}$

## VI. Problem Solving Through Equations

### A. Algebraic representation

1. Review II-B-3

2. Have pupil express the following:

   a. Tom is 14 years old. How old was he 5 years ago? How old will he be 3 years from now?

   b. If we do not know Sam's age, we can let a = the number of years in Sam's age.

   c. Sam's age 3 years from now. (a + 3)

   d. Sam's age 5 years ago. (a - 5)

   e. The length of a room is 15 ft. The width is 5 ft. less than the length. What is the width? Draw and label diagram.

   f. The length of a room ix x ft. Express the width of a room if it is 5 ft. less than the length. Draw and label diagram.

   g. You have 4 nickels in your pocket. Jack has 2 nickels more than you. How many nickels has Jack?

   h. You have a certain number of nickels in your pocket. Jack has 2 more than you. Express the number of nickels Jack has.

   i. How many feet are there in 1 yard? 3 yards? 5 yards? y yards?

   j. How many inches are there in 1 foot? in 2 feet? in 4 feet? in c feet?

   k. How many cents in one dollar? in 3 dollars? in 4 dollars? in g dollars?

   l. How many cents in one nickel? in 2 nickels? in 6 nickels? in n nickels?

3. Have pupil state the following expressions algebraically:

a. A train travels 30 miles per hour. How far does it go in 2 hours? in 3 hours?

b. A boy on a bicycle travels 5 mph. His friend travels 3 mph faster. What is his friend's rate?

c. A boy on a bicycle travels r miles per hour. Express the rate of an automobile which is traveling 30 miles per hour faster than the bicycle.

d. A rectangle is 6 feet long. The width is 2 feet less than the length. What is the width? Draw and label diagram.

e. A rectangle is f feet long. Express the width which is 3 feet shorter than the length. Draw and label diagram.

f. The side of a square is 6 inches in length. What is the perimeter? Draw and label diagram.

g. The side of a square is s inches in length. What is the perimeter? Draw and label diagram.

h. The radius of a circle is 3 inches in length. What is the diameter? Draw and label diagram.

i. The radius of a circle is r feet in length. What is the diameter? Draw and label diagram.

j. If one number is 7, what is the next whole number? the next?

k. If one number is n, what is the next whole number? (consecutive number)

4. Identifying and writing open sentences

a. Have pupil select the sentences which express the following:

1) A number increased by 5 is 8.

$n - 5 = 8$ $\qquad$ $n + 5 = 8$ $\qquad$ $5n = 8$ $\qquad$ $\frac{1}{5}n = 8$

2) A boy is 10 years old. He is $\frac{1}{4}$ as old as his father.

$a - \frac{1}{4} = 10$ $\qquad$ $a + \frac{1}{4} = 10$ $\qquad$ $4a = 10$ $\qquad$ $\frac{1}{4}a = 10$

3) A certain number of pencils cost 56¢. The price of 1 pencil is 7¢.

$n - 7 = 56$ $\qquad$ $n + 7 = 56$ $\qquad$ $7n = 56$ $\qquad$ $\frac{n}{7} = 56$

4) The perimeter of a square whose side is s is 60 inches.

$s - 4 = 60$ $\frac{s}{4} = 60$ $4s = 60$ $s + 4 = 60$

5) The area of the floor of a room whose width is 12 feet is 180 square feet.

$b - 12 = 180$ $12b = 180$ $b + 12 = 180$ $\frac{b}{12} = 180$

B. Suggested practice

1. Have pupil practice with problems which can be solved in one operation. Have pupil solve and check the equations in 4-a above, 1) through 5).

   Note: Solving "word" problems requires several skills. Skill in reading and interpretation, skill in translating the English sentence into an equivalent open sentence (equation), and skill in solving the resulting equation. While the one step problems may be solved easily by arithmetic, it is suggested that pupils be taught to solve these by equations in preparation for more difficult problem work. The degree of difficulty that can be effectively taught will vary with the ability of the class.

   The usual steps in solving "word" problems:

   Careful reading and visualization of the relationship found in problem.

   Identify what is known and what is to be found.

   Writing the open sentence (equation) which expresses the meaning of the problem.

   Solving the equation by the application of the appropriate principles.

   Checking the answers in the words of the problem.

2. Have pupils write the appropriate equation for each of the following. Solve and check.

   a. If 5 is added to a certain number, the sum is 127. What is the number? What are we asked to find? Therefore,

      Let $x$ = the number
      $x + 5 = 127$

b. Seven times a certain number is 91. Find the number.

c. Twelve times Bill's age is 168. How old is Bill?

d. One half of a number is 121. What is the number?

e. A team lost 6 games which is $\frac{1}{5}$ of all games **played.** How many games have been played?

f. John worked 8 hours and received $10 for his work. How much did he earn in one hour?

g. If 127 is added to a number, the result is 342. What is the number?

h. Five years ago Ben was 14 years old. How old is he today?

i. If 11 is subtracted from a number, the result is 134. What is the number?

j. A man made a payment of $755 on his home and still owed $2575. How much did he owe before the payment?

3. In the following problems, have the pupils

Read the problem carefully.
Identify the number represented by the letter.
Write the equation.
Solve.
Check.

a. 4 pupils, or $\frac{1}{8}$ of the class, did not join the G.O. How many pupils are there in the class?

b. A boy can ride a bicycle at the rate of 6 miles per hour. How long would it take to ride 30.6 miles?

c. Sam earned $5.24 more than his brother. Sam earned $68.75. How much did his brother earn?

d. A pilot covered a distance of 1440 miles in 8 hours. What was his average rate per hour?

e. Mae's father owed $50 for a record player. This is to be paid in 12 equal monthly payments. How much did he pay each month?

f. The sum of the three angles of a triangle is 180°. If one angle is 20°, a second is 35°, how many degrees does the third angle contain?

## VII. Solving Equations

### A. Suggested procedure for solving equations of the type $\frac{a}{b}x = c$

1. Review with the pupil the solution of equations of the type $2x = 6$. (See V-3-c-1) Have pupil recall that in solving these equations, he divided both members of the equation by the coefficient of x.

2. Suggest the equation: $\frac{3}{4}n = 12$

   Lead the pupil to see that the coefficient of n in this equation is the fraction $\frac{3}{4}$. Have pupil recall the method used to divide by a fraction.

   For example,

   $6 \div \frac{1}{3} = 6 \times \frac{3}{1}$ $8 \div \frac{2}{3} = 8 \times \frac{3}{2}$

3. Guide pupil to realize that he must divide both sides of the equation by $\frac{3}{4}$.

   $\frac{3}{4}n = 12$

   $\frac{\frac{3}{4}n}{\frac{3}{4}} = 12 \div \frac{3}{4}$

   $n = 12 \times \frac{4}{3}$

   $n = 16$ 16 is the root of the equation.

   Note: In those classes where pupil is unsure of division of fractions, this topic may be reviewed here. See Grade 7 bulletin.

4. Check: If 16 is the root of the equation, when we replace the symbol n with 16, the statement will be true.

   $\frac{3}{4}n = 12$

   $\frac{3}{4} \times 16 \overset{?}{=} 12$

   $3 \times 4 \overset{?}{=} 12$

   $12 = 12$

5. Suggested practice

a. $\frac{1}{4}d = 4$
b. $\frac{2}{3}n = 14$
c. $\frac{3}{5}a = 20$
d. $\frac{5}{6}y = 15$
e. $\frac{3}{4}b = 21.6$
f. $\frac{4}{5}x = 20.4$
g. $\frac{1}{2}a = 3\frac{1}{2}$
h. $\frac{3}{4}a = 3\ \frac{1}{2}$

## B. Solving two-step equations

### 1. Solving equations by subtraction and division

a. Have pupils review solution of equations such as: $x + 7 = 12$. (See IV-A-1-a)

Have pupil recall how he solved equations by subtraction to get an equivalent equation in which x was the only member on the left. (See example above)

b. Have pupil review solution of equations such as $3x = 15$. (See V-A-3-c) Have pupil recall how he solved equations by dividing to get an equivalent equation in which x was the only member on the left.

c. Have pupil generalize that in solving equations in which x was alone as one member of the equation, (i.e., $x = 3$), the solution of this equation was obvious.

d. Pose the problem: I am thinking of a number. If I multiply the number by 3 and add 7, my answer is 22. What is the number? Have pupils do this intuitively. Elicit from pupil how he arrived at the solution.

1) Have pupil try any number he chooses (the guess method) e.g. 9,

then $3 \times 9 + 7 = 34$ 9 is too large
try 5, then $3\times5+7 = 22$ 5 is the number

or, I added 7 to get to 22, so I'll subtract 7 from 22 and get 15.

I multiplied by 3 to get to 15, so I'll divide 15 by 3 and get 5.

That was the number with which I started.

2) Have pupil write the equation for the problem.

Suggested procedure: What are we asked to find? (the number)
Therefore: Let x = number
$3\,x + 7 = 22$

Since we wish to arrive at an equivalent equation in the form x = ?, we will first try to eliminate the "+7". How?

| | |
|---|---|
| $3x + 7 - 7 = 22 - 7$ | Subtract 7 from both members of the equation |
| $3x = 15$ | Equivalent equation |
| $\frac{3x}{3} = \frac{15}{3}$ | Divide both members by 3 |
| $x = 5$ | Equivalent equation |

Therefore, 5 is the root (the number we were to find).

Check: Since this problem was presented in words, we formed the equation. The equation, therefore, may not be correct.

To determine if our answer is the answer to the problem, we must check in the words of the problem.

5 multiplied by 3 = 15 and 15 + 7 = 22

Note to teacher: Emphasize the difference in the form of check used in examples such as in e (below) and those in which the problem is stated in words and the equation formed by the pupils. If the equation is incorrectly formed, yet all subsequent work based on the incorrect equation is correct, the answer will not be the answer to the problem, although it will check in the equation.

e. Practice: Have pupil solve and check the following equations using the above form. Steps a) through d) of 1) and 2).

1) $2x + 3 = 7$
2) $4n + 3 = 11$
3) $21 = 5n + 4$
4) $3y + .2 = 9.2$
5) $7a + 2 = 16.7$
6) $2b + 1\frac{1}{2} = 5\frac{1}{2}$
7) $.04p + 2 = 2.08$
8) $3x + 1\frac{1}{4} = 2\frac{3}{4}$

2. Solving equations by addition and division

a. Have pupil solve equations such as x - 5 = 4.(See IV-A-3-a-b)

b. Have pupil solve this equation:

| | |
|---|---|
| $3x - 4 = 2$ | |
| $3x - 4 + 4 = 2 + 4$ | Adding 4 to both members |
| $3x = 6$ | Equivalent equation |
| $\frac{3x}{3} = \frac{6}{3}$ | Dividing both members by 3 |
| $x = 2$ | Equivalent equation |

The root is 2.

c. Check: Replacing x by 2

$3 \times 2 - 4 \overset{?}{=} 2$

$6 - 4 \overset{?}{=} 2$

$2 = 2$

At this point introduce the use of parentheses () for multiplication instead of the multiplication sign which may be confused with "x". For example, the first equation above may be written, $3(2) - 4 = 2$.

d. Practice - Solve and check:

1) $2x - 3 = 5$
2) $3a - 2 = 13$
3) $4x - \frac{1}{2} = 7\frac{1}{2}$
4) $5y - .2 = 10.3$
5) $3x - \frac{1}{4} = 8$
6) $2b - 1.2 = 3.6$
7) $6c - 2 = 5.2$
8) $12x - 23\frac{1}{2} = 72\frac{1}{2}$

3. Solving equations by addition, subtraction, and multiplication

a. Have pupil recall that he solved equations such as $\frac{k}{3} = 4$.

| | |
|---|---|
| $\frac{k}{3} = 4$ | |
| $3 \times \frac{k}{3} = 3 \times 4$ | Multiply both members by 3 |
| $x = 12$ | Equivalent equation (See V-4-a-b) |

The root is 12.

b. Have pupil solve

| | |
|---|---|
| $\frac{x}{3} + 2 = 6$ | |
| $\frac{x}{3} + 2 - 2 = 6 - 2$ | Subtracting 2 from both members |
| $\frac{x}{3} = 4$ | Equivalent equation |
| $3 \times \frac{x}{3} = 3 \times 4$ | Multiplying both members by 3 |
| $x = 12$ | Equivalent equation |

The root is 12.
Check:

$\frac{12}{3} + 2 \overset{?}{=} 6$

$4 + 2 \overset{?}{=} 6$

$6 = 6$

c. Have pupil solve:

| | |
|---|---|
| $\frac{x}{3} - 2 = 2$ | |
| $\frac{x}{3} - 2 + 2 = 2 + 2$ | Adding 2 to both members |
| $\frac{x}{3} = 4$ | Equivalent equation |
| $3 \times \frac{x}{3} = 3 \times 4$ | Multiplying both members by 3 |
| $x = 12$ | Equivalent equation |

The root is 12. Check as above in 3-b.

4. Practice - Solve and check

a. $\frac{x}{2} - 3 = 1$

b. $\frac{1}{4}g + 2\frac{1}{2} = 4$

c. $\frac{n}{15} - 2 = \frac{2}{5}$

d. $\frac{x}{7} - 4 = 4.9$

e. $\frac{c}{3} + 3 = 1.6$

f. $\frac{n}{.3} + 2 = 6.7$

g. $\frac{x}{2.5} - 3 = 4.4$

h. $\frac{y}{2} + .3 = 1.7$

C. Solving equations of the type $ax \pm bx = c$

1. The distributive principle

a. Have pupil consider how David, a pupil in class 9-2, might keep a record of the money he collects in his homeroom. On Monday, 5 people gave him 10¢ each, and on Tuesday, 4 people gave him 10¢ each. How much money did he collect? He figures,

$5 \times 10 + 4 \times 10$ or $50 + 40$ or 90¢. He collected 90¢.

b. Have pupil consider that:

David could have kept his records in another way. Since everyone gave him the same amount, 10¢, he could write the number of people who paid him on Monday, and the number of people who paid him on Tuesday. He would then multiply the total number by 10. Thus, his figuring looks like this:

$(5+4) \times 10$ Note: The parentheses indicate that the quantities within them are to be treated as a unit, i.e., 9.

The final result is the same in both ways of keeping records, and so

(5x10) + (4x10) = (5+4) x 10

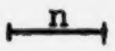
5 tens + 4 tens = 9 tens or 90

Have pupil see that when we add 5 and 4 and then multiply the 9 by 10, we get the same result as when we multiply both 5 by 10 and 4 by 10 and add the products.

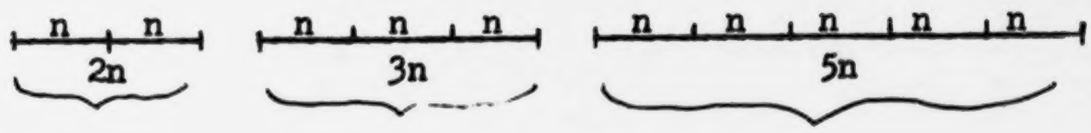
c. Geometric illustration of distributive principle

| 1) 2 x 3 | + | 4 x 3 | = | 6 x 3 |
|---|---|---|---|---|
| 0 0 0 | | 0 0 0 | | 0 0 0 |
| 0 0 0 | | 0 0 0 | | 0 0 0 |
| | | 0 0 0 | | 0 0 0 |
| | | 0 0 0 | | 0 0 0 |
| | | | | 0 0 0 |
| | | | | 0 0 0 |

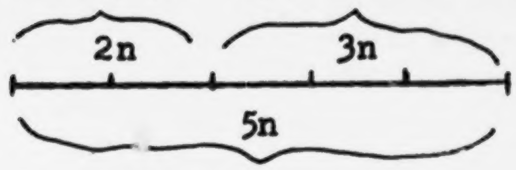
2) Let n represent the number of units of length in a given line segment.



Have pupil represent 2n, 3n, 5n



Have pupil represent 2n + 3n.



2n + 3n = 5n

d. Have pupil test the truth of sentences such as the following:

1) 6 tens and 2 tens = 8 tens Answer: 60 + 20 = 80

5 threes - 2 threes = 3 threes

(4x3) + (2x3) = 6x3

(9x4) - (6x4) = 3x4

2) Have pupil complete sentences such as the following:

a) $(2x5) + (7x5) = \_\_\_\_ \times 5$

b) $(10x2) - (5x2) = \_\_\_\_ \times 2$

c) $(\_\_\_\_ \times 3) + (6x3) = 10 \times 3$

d) $(5x7) + (\_\_\_\_ \times 7) = 9 \times 7$

e) $(2 \times \_\_\_\_) + (4 \times \_\_\_\_) = 6 \times \_\_\_\_$

2. Using the distributive principle

a. Have pupil see that just as $(4x2) + (3x2)$ may be written as $(4+3) \times 2$ or $7 \times 2$, so $4x + 3x$ may be written as $(4+3)x$ or $7x$.

b. Have pupil see that just as $(9x5) - (7x5)$ may be written as $(9-7) \times 5$, or $2 \times 5$, so $9x - 7x$ may be written as $(9-7)x$ or $2x$.

c. Have pupil write the following in simpler form:

$2(7) + 3(7)$ Note: $2(7)$ is another way of writing $2 \times 7$.

$4(12) - 2(12)$

$5a + 2a$

$9y - 3y$

3. Using the distributive principle in solving equations

a. Have pupil consider an equation such as:

$2x + 3x = 10$ Equation

Help pupil to see that the left member may be written in simpler form $5x$ (because of the distributive principle), and the equation may therefore be written as:

$(2+3)x = 10$

$5x = 10$

$\frac{5x}{5} = \frac{10}{5}$ Divide both members by 5

$x = 2$ Equivalent equation in which the root, 2, is obvious

The root is 2.

Check: Have pupil check 2 as the root of the equation $2x + 3x = 1$

b. Have pupil consider an equation such as:

| | |
|---|---|
| $7y - 4y = 15$ | Equation |
| $(7-4)y = 15$ | |
| $3y = 15$ | Distributive principle |
| $\frac{3y}{3} = \frac{15}{3}$ | Divide both members by 3 |
| $y = 5$ | Equivalent equation in which the root, 5, is obvious. |

4. Suggested practice

Solve the equations. State the reason for each step. Check your answers.

a. $5m + 3m = 16$

b. $x + 4x = 20$

c. $3x + 7x = 20$

d. $24 = 10y - 2y$

e. $\frac{3}{2}x + \frac{1}{2}x = 4$

f. $\frac{4}{5}y - \frac{3}{5}y = 3$

g. $6 = 1.5a + .5a$

h. $7.1x - 2.1x = 15$

i. $x + 3x + 4x = 32$

5. Verbal problems

Note: Have pupil solve problems with arithmetic numbers before using algebraic representation.

a. The number of boys in Class 9-2 is twice the number of girls. There are 10 girls in the class. How many boys are there. What is the total number of pupils?

If the number of girls is represented by x, how would you express the number of boys? Express the total number of pupils.

b. The number of boys in Class 9-4 is three times the number of girls. The total number of pupils is 28. How many girls are in the class?

Illustrative solution: Let $x$ = number of girls
Then $3x$ = number of boys

$x + 3x = 28$ (Total number of pupils is 28)
$4x = 28$ (Distributive principle)
$x = 7$ (Equivalent equation in which the root is obvious.)

There are 7 girls in the class. Check results with conditions in problem. If there are 7 girls in the class, there are 3x7 or 21 boys in the class. Total number: 7+21 or 28.

c. A boy has a board which he cuts into two pieces. One piece is 10 inches long; the other is four times as long. How long is the larger piece? What is the total length of the two boards?

If in example c. above, the number of inches in the shorter piece is represented by x, how would you represent the length of the longer piece? Represent the total length. Draw and label diagram.

d. A board which is 63 inches long is to be cut into two pieces so that one will be 8 times as long as the other. How long should the pieces be?

Suggested procedure:

What are we asked to find? (length of the two pieces)

Can we represent the length of both pieces by the same letter? (No, one is larger than the other.)

Tell pupils that it is usually easier to use the letter to represent the number we know least about. Which length do we know least about? (smaller piece)

Therefore:

Let $x$ = number of inches in smaller piece
Then $8x$ = number of inches in larger piece

$x + 8x = 63$
$9x = 63$ Distributive principle
$\frac{9x}{9} = \frac{63}{9}$ Divide both sides by 9
$x = 7$ Equivalent equation in which root is obvious.

Then $8x = 56$

One piece is 7 inches long and the other is 56 inches long.

Check: Two conditions must be checked:

1. Is one piece 8 times the other? Explain.
2. Is the sum of the two pieces 63? Explain.

e. One number is 5 times another. The smaller number is 8. What is the larger? What is the difference between the two numbers?

If the smaller number is represented by n, how would you represent the larger number? How would you represent their difference?

f. One number is 6 times another. Their difference is 35. Find the numbers.

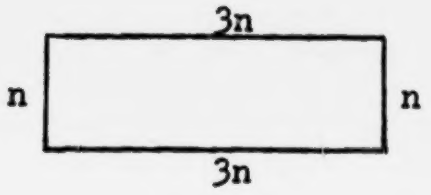
g. The length of a rectangular table is twice its width. If the table is 12 inches wide, how long is it? What is the perimeter? Diagram.

If the table is x inches wide, how would you express the length? Represent the perimeter. Diagram.

h. The length of a rectangular table is 3 times its width. The perimeter of the rectangle is 88 inches. Find the length and the width of the table. Diagram.

Illustrative solution: Let n be the number of inches in the width of the rectangle. Then 3n is the number of inches in the length.

Encourage pupil to draw a picture to help analyze and solve the problem:



$$\begin{aligned} n + 3n + n + 3n &= 88 && \text{(the perimeter is 88 inches)} \\ 8n &= 88 && \text{(distributive principle)} \\ n &= 11 && \text{(equation in which the root is obvious)} \\ \text{Then } 3n &= 33 \end{aligned}$$

The table is 11 inches wide and 33 inches long.

Checking: The length is 3 times the width because $33 = 3 \times 11$.
The perimeter is $11 + 33 + 11 + 33 = 88$.

i. Tom and Bob together weigh 240 pounds. Tom weighs twice as much as Bob. What does each weigh?

j. Eight times some number less 3 times the same number equals 30. What is the number?

k. Joe and John earned $7.50 selling magazines. They had agreed that Joe was to receive 1.5 times the amount George was to receive. How much did each receive?

## VIII. Using Equations to Solve Verbal Problems (more than one step)

### A. Symbolic representation and illustrative solutions

1. Review translation of verbal problems into equations (one step) see VI.

2. Analysis of problems involving types of equations developed in VI and VII.

Note: It is suggested that the illustrative solutions given in VIII-A-2 be used when the particular type of problem is taught. Develop each type by means of a problem in arithmetic first in "Suggested practice" (VIII-B)

Number problems #1-6
Consecutive number problems #7-9
Coin problems #10-12
Geometry problems #13-16
Motion problems #17-21

a. Number problems

1) Multiply 6 by 2. Add 3. What is the total?

2) Select a number x. Multiply it by 2. Add 3. Express the total in terms of x.

3) Problem: A number is multiplied by 2. Three is added to it giving a total of 15. What is the number?

Solution: let $x$ = the number

$$\begin{array}{ll}
2x & \text{(2 times the number)} \\
2x + 3 & \text{(3 added to two times the number)} \\
2x + 3 = 15 & \text{(the total is 15)} \\
2x + 3 - 3 = 15 - 3 & \text{(subtract 3 from both members)} \\
2x = 12 & \text{(equivalent equation)} \\
\frac{2x}{2} = \frac{12}{2} & \text{(divide both members by 2)} \\
x = 6 & \text{(equivalent equation)}
\end{array}$$

The number is 6.

4) Check in problem:

The number 6 x 2 = 12. Three added to 12 is 12 + 3 = 15. True.

b. Consecutive number problems

1) Develop with pupil the meaning of consecutive numbers. (When a whole number is increased by one, it is the consecutive number of the first, e.g., 13 is the next higher consecutive number of 12.)

2) 9 is the first of three consecutive numbers. What are the numbers?

3) What is the sum of 7 and its next higher consecutive number?

4) If x represents one number, express in terms of x the next consecutive number. Express their sum.

5) The sum of two consecutive numbers is 27. What are the numbers?

Illustrative solution:

Let $x$ = the first number

$x + 1$ = the consecutive number

$x + x + 1 = 27$ the sum of the numbers

$$2x + 1 = 27$$

$$2x = 26$$

$$x = 13$$

One number is 13, the other is 14.

6) Check in problem:

The two numbers 13 and 14 are consecutive numbers. Their sum is 27.

Note: 2 conditions in check.

c. Coin problems

1) What is the value in cents of 5 dimes? of 7 dimes? of 10 dimes?

2) To find the value in cents of any number of dimes, we multiply the number of dimes by _____.

3) Sally has x dimes. What is their value in cents?

4) What is the value in cents of 6 dimes and 3 pennies? of 9 dimes and 7 pennies? of 4 dimes and 7 pennies? of x dimes and 2 pennies?

5) Peggy has 3 dimes and 5 more pennies than she has dimes. How many pennies has she? What is the total value (in cents) of her dimes and pennies?

6) Complete the following table.

| Number of Dimes | Number of Pennies | Total Value in Cents |
|---|---|---|
| 2 | 0 | |
| 1 | 5 | |
| 7 | 11 | |
| 12 | 3 | |
| x | 0 | |
| x | 2 | |
| | x+6 | |

7) John has x dimes. What is their value in cents? He has 3 more pennies than dimes. How many pennies has he? Express in terms of x the total value (in cents) of his pennies and dimes.

8) Present the problem: Lois has a number of dimes. She has 6 pennies more than she has dimes. She has 83¢ altogether. How many dimes has she? How many pennies?

Illustrative solution:

Let $x$ = number of dimes she has
$10x$ = value of the dimes in cents
$x + 6$ = number and value of pennies Lois has
$10x + x + 6 = 83$ the value of the dimes and pennies in cents is 83
$11x + 6 = 83$
$11x = 77$
$x = 7$

Lois has 7 dimes and 13 pennies.

9) Check in problem: If Lois has 7 dimes, then she has 7+6=13 pennies. 7 dimes is 70¢ + 13 pennies = 83¢.

Note: Similar problems involving any other coin (nickels, quarters) combined with pennies may be given.

d. Geometry problems - Draw and label a diagram for each problem.

1) Review meaning of perimeter; perimeter of a triangle, square, rectangle. Sum of angles of a triangle.

2) One angle of a triangle is 30°; a second angle is 60°. What is the size of the third angle?

3) One angle of a triangle is x degrees, a second is twice as large and the third is three times the size of the first. Express the size of the second and the third angles in terms of x.

4) In a triangle one angle is twice as large as another. The third is three times as large as the smallest. What is the size of each angle?

Illustrative solution:

Let $x$ = the number of degrees in the smallest angle
$2x$ = the number of degrees in the second angle
$3x$ = the number of degrees in the third angle

$x + 2x + 3x = 180$ the sum of the angles of a triangle
$6x = 180$
$x = 30$

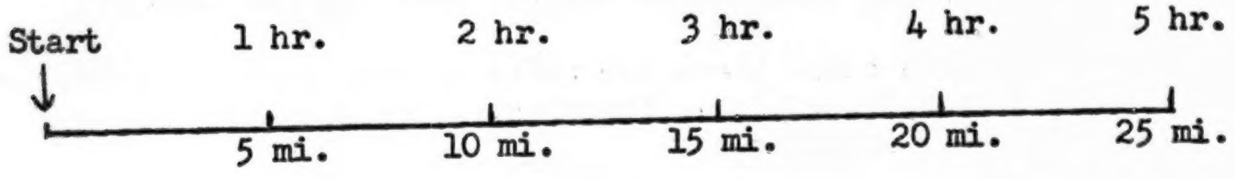
The smallest angle is 30°, the angle which is twice as large is 60° and the angle which is three times as large is 90°.

5) Check in problem: If one angle is 30°, the second is 60° and the third is 90°, the sum of the angles is 180°.

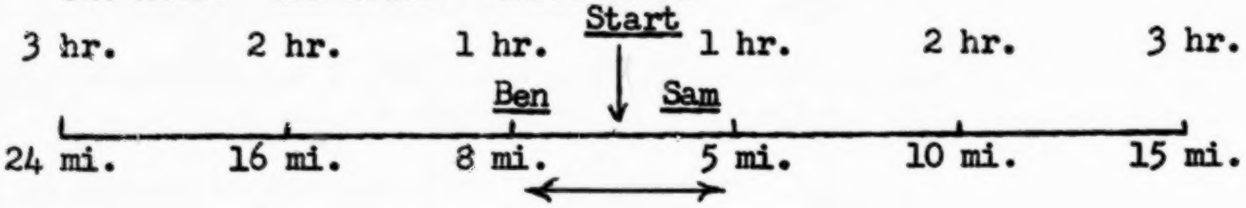
e. Motion problems

1) Review with pupils the distance formula: $d = rt$, the meaning of d, of r, of t.

2) Sam starts from the library and rides at 5 miles per hour. How far has he gone in one hour? in two hours? in three hours?



3) Ben starts from the library at the same time as Sam and rides in an opposite direction from Sam at the rate of 8 mph. How far apart are Sam and Ben at the end of one hour? two hours? three hours?



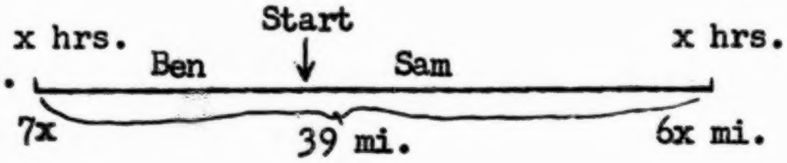
4) Sam and Ben on bicycles start from the library at the same time and ride in opposite directions, Sam at 6 mph and Ben at 7 mph. In how many hours will they be 39 miles apart?

Illustrative solution:

Let $x$ = the number of hours they ride
$6x$ = the number of miles Sam rides
$7x$ = the number of miles Ben rides
$6x + 7x = 39$ (after x hours the boys will be 39 miles apart)

$13x = 39$
$x = 3$

The boys each rode 3 hours.



Check in problem: Sam travels 18 miles (6 miles an hour for 3 hours = 18) and Ben travels 21 miles (7 miles an hour for 3 hours = 21). Traveling in opposite directions, they are 39 (18+21) miles apart.

B. Suggested practice

1. Four times a number is increased by 6. The result is 64. What is the number?
2. The larger of two numbers is 19 more than the smaller number. The sum of the numbers is 75. What are the numbers?
3. In a class there are twice as many girls as there are boys. There are 39 pupils in the class. How many boys are there in the class? How many girls?
4. As a baby sitter, Catherine is paid 75¢ an hour, plus 30¢ for carfare. One evening she received $4.05. How many hours did she baby sit?
5. A bottle and a cork cost $1.10. If the bottle costs $1 more than the cork, what is the cost of each? (5¢; $1.05)

6. Peter is twice as old as his baby sister. Their father is 16 times as old as the baby. The sum of their ages is 38. How old is the baby? How old is Peter? How old is Peter's father?

7. The sum of two consecutive numbers is 89. What are the numbers?

8. The sum of two consecutive numbers is 49. What are the numbers?

9. If there are 3 consecutive numbers, such that the sum of the first and third is 28, what are the numbers?

10. Harry's piggy bank contains only nickels and pennies. There are 3 times as many nickels as pennies. How many pennies has he if the total amount of money is \$3.36? (Use p for pennies)

11. Peter has twice as many dimes as pennies in his pocket. The total he has is \$1.47. How many pennies and dimes has he?

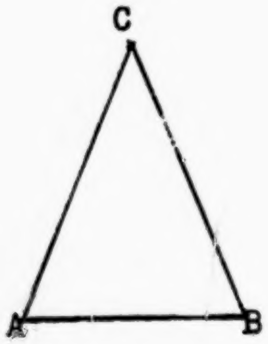
12. Leonard has a number of nickels in his coin jar. He has 3 more pennies than he has nickels. He has 99¢ altogether. How many nickels has he? How many pennies?

13. The length of a rectangle is 4 times the width. The perimeter is 10 inches. What are the dimensions?

14. The sum of two angles is $90^\circ$. If one angle is 4 times as large as the other, how many degrees are there in each angle?

15. In the isosceles triangle ABC, the equal sides are each 4 inches longer than the base. The perimeter is 38 inches. How many inches does the base measure? How long is each side?

16. In triangle ABC, $\angle A = \angle B$ and $\angle C$ is $30^\circ$ less than either of the other two. What is the size of each angle?



17. Two automobiles start from the same place and travel in opposite directions. One averages 45 mph, the other 60 mph. In how many hours will they be 840 miles apart?

18. Two boys walk toward each other from points $10\frac{1}{2}$ miles apart, one at 3 mph, the other at 4 mph. In how many hours will they meet?

19. Dennis leaves New York City for Albany 150 miles away, and travels at the rate of 35 miles an hour. Sam leaves Albany for New York City the same time, traveling at the rate of 40 miles an hour. In how many hours will they meet if they travel along the same highway?

20. Two cities, A and B, are 380 miles apart. An automobile leaves City A going toward City B at the rate of 50 miles per hour. At the same time, a second car, at the rate of 45 miles per hour, leaves City B for City A. How soon will the two cars pass each other?

21. How far has each car traveled in problem 20?

# II.

# DIRECT MEASUREMENT

## I. Standard Measuring Unit

### A. Suggested procedure

1. Have pupil learn the meaning of measurement of length as a comparison of an unknown length with a unit length.

   a. Have pupil measure the width of his desk using the length of his pencil as a unit length.

   Note: The unit length (pencil) will probably be different for each student.

   1) Record pupil report of measurement of desk in table form such as:

| Row | Seat | Number of Pencil Lengths |
|---|---|---|
| 1 | 1 | about $7\frac{1}{3}$ |
| 2 | 1 | about $9\frac{1}{4}$ |
| 3 | 1 | about $11\frac{1}{4}$ |
| 4 | 1 | about 5 |
| 5 | 1 | about 23 |
| 6 | 1 | about $8\frac{1}{2}$ |

   2) Have pupil consider the reasons for the different results.

   3) From his observation (measurement) and discussion, have pupil conclude:

      a) the width of the desk did not change
      b) the lengths of the units (pencil lengths) differed
      c) therefore, measurements with different pencil lengths are unsatisfactory.

   b. Have several pupils measure the width of the room using their foot length as a unit.

   Note: Make pupil aware of the fact he is using a unit based upon a part of the body.

   1) Record pupil report of measurement of width of room in table form as in I-A-1-a-1).

2) Have pupil examine measurements: a boy with a large foot, 25 shoe lengths; a boy with a shorter foot, 27 shoe lengths; a girl, 32½ shoe lengths. (set up table as above)

3) Help pupil conclude:

   a) the width of the room did not change
   b) the lengths of the unit (foot lengths) differed for individuals
   c) therefore, the measurements with the different foot lengths are unsatisfactory.

4) Have pupil discuss this question: Why was there a greater variation in the desk measurements than in the width of the room measurements?

   a) the pencil lengths varied greatly
   b) the foot lengths were more nearly the same.

5) Have the pupil consider which is the more satisfactory unit, the pencil length or the shoe length? He should be led to realize:

   a) the variation of the shoe lengths in his class is less than the variation in the pencil lengths
   b) the shoe lengths are more nearly uniform (equal) than are the lengths of pencils
   c) the shoe length is a more satisfactory unit.

6) Have pupil consider the problem of communicating measurements using his shoe length as a unit. Have pupils realize the need for standardizing the unit.

c. Have several groups of pupils measure the width of the room using blackboard pointers as a unit. (All blackboard pointers are the same length.)

1) Record pupil report of measurement of width of room in table form as in I-A-1-a.

2) Have pupil examine measurements.

3) Have pupil conclude that the results all seem to agree.

4) Have pupil consider the problem of communicating measurements in which pencil lengths, shoe lengths, and length of blackboard pointer are units of length. He should realize the blackboard pointer, of uniform length, is the best unit to use in telling of the width of the room.

2. Have pupil learn what is meant by a standard unit of length and the need for such a unit.

   Note: Many commercial organizations issue free charts, pamphlets, etc. dealing with the history of measurement, nature of approximation and precision. A partial list follows:

   1) Ford Motor Co., Dearborn, Michigan
      a) Charts on History of Measurement
      b) Booklets, History of Measurement

   2) General Motors Company, Detroit, Michigan
      a) Booklet, Precision

   3) Chrysler Motors Corp., Detroit, Michigan
      a) Booklet, Problems from Industry

   4) Starrett Tool Company

   5) Lufkin Rule Co.

   a. Tell the story of David and Goliath and give Goliath's height. (6 cubits and a span) Can we indicate a point on the wall which shows Goliath's height? (9 ft., 9 in.) (cubit is about $1\frac{1}{2}$ ft; span is about 9 in.)

   b. Have pupil see that in order to determine Goliath's height, he needs to know the meaning of "cubit" and "span."

   c. Discuss the historical background of these and other units for measuring lengths (see any encyclopedia).

   d. Have pupil trace and cut out, and use his cubit (length from his elbow to tip of middle finger) to measure such lengths as length of teacher's desk, width of blackboard, and width of door.

   e. Have him compare the results of his measurement with those of one or two of his classmates.

   f. Have him discuss the reason for the difference in results (have him compare his cubit with that of one or two of his classmates).

   g. Have him conclude that the cubit is a personal unit of measure and its length depends on the size of the pupil.

   h. In a similar manner, have pupil study the idea of a span = ½ cub the need for a smaller unit will have come up from results obtained in I-A-2-d.

i. Have pupil discuss what he might do so that if he reports a measurement to be two cubits, it will mean the same length to everyone. The pupil should be led to realize the need for a standard length to be called a cubit.

j. Have pupil report on how some of the measures he uses (the English system) originated and how they were standardized.

For example, he might consider shoe size.

1) Have pupil who wears size 10 measure his shoe. He will find his shoe length is 12" (approximately). Have him consider how it happens that a size 10 shoe measures 12". Tell him:

   a) King Edward II in 1324, prompted by the need for standardizing the many and varied units of length used in his country, decreed that 3 barleycorns placed end-to-end would be equal to one inch.

   b) The King had the left foot of **some men in his kingdom** measured. The largest foot measured 39 barleycorns, i.e., 13 inches, and it was decided to call it size 13.

| Number of barleycorns | Shoe size | Length of shoes |
|---|---|---|
| 39 | 13 | 13" |
| 38 | 12 | $12\frac{2}{3}$" |
| 37 | 11 | $12\frac{1}{3}$" |
| 36 | 10 | 12" |
| 35 | 9 | $11\frac{2}{3}$" |
| 34 | 8 | $11\frac{1}{3}$" |
| 33 | 7 | 11" |
| 32 | 6 | $10\frac{2}{3}$" |
| 31 | 5 | $10\frac{1}{3}$" |

1 inch = 3 barleycorns

$\frac{1}{3}$ inch = 1 shoe size

B. Suggested practice

1. Have pupil use length of book as a unit of measure. Have him find the width of door, desk, and blackboard. (Use after I-A-1-b)

2. How many spans are there in 2 cubits? 5 cubits? What part of a cubit is a span?

3. If Noah's Ark was 300 cubits long, what would it measure in feet?

4. The Romans used a double step or pace of about 5 feet as a unit and 1000 of these paces was called a mile. How many feet were there in a Roman mile? How many feet are there in a mile today? How much longer is our mile than the Roman mile?

5. Have pupil answer the following:

   a. 1 yard is about _____ cubits.

   b. 1 yard is about _____ spans.

   c. 1 yard is _____ feet.

   d. 1 yard is _____ inches.

   e. 1 foot is _____ inches.

   f. 1 mile is _____ feet.

   g. 1 mile is _____ yards.

   h. 1 mile is _____ inches.

II. Reading a Ruler

A. Suggested procedure

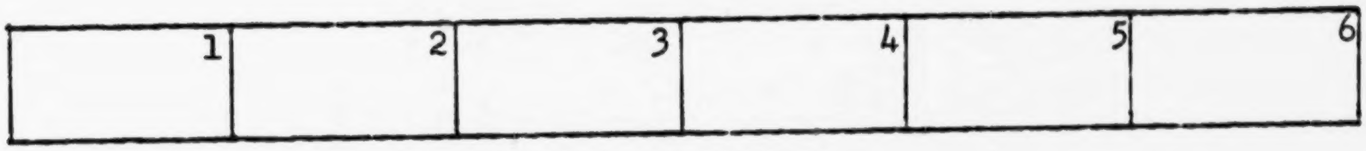
1. Have pupil discuss instruments for measuring length. Bring to class such instruments as ruler, steel tape, yardstick, carpenter's rule, architect's and engineer's scale and micrometer (or pictures of the instruments).

2. Have pupil construct his own 6" ruler divided into inches only. (Supply pupil with a strip of oaktag 6" long and a piece 1" long.)

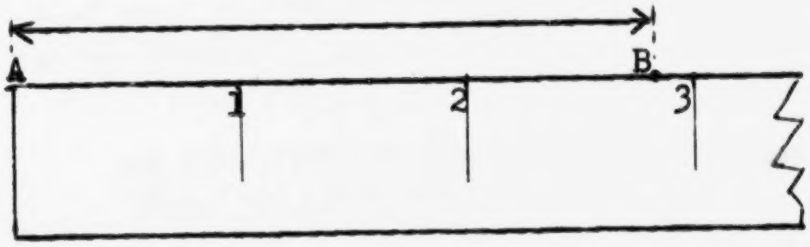
   a. Have him mark off the unit length on the 6" length.

   b. Have pupil understand that the distance from the left edge of the ruler to the first line represents 1 inch. Similarly, the distance from the line marked 1 to the line marked 2 is also 1 inch, and so on.

1) What is the distance from 5 to 6?
2) What is the distance from 1 to 3? 2 to 4? 0 to 2?
3) Name any two numbers which are 2 inches apart; 3 inches apart.
4) What number is understood at the extreme left edge of a ruler?



c. Have pupil find the distance from A to B.



The unit of measure is 1" and by counting the pupil finds the distance from A to B to contain the unit approximately 3 times (the measurement is nearer 3" than 2").

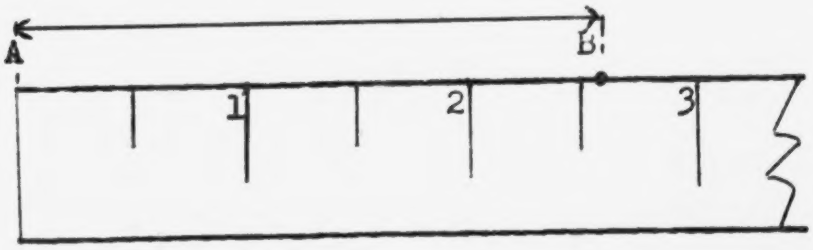
d. Have him use his pupil-made ruler to

1) measure various lengths and
2) draw **line segments of specific lengths**

3. Have pupil refer to II-A-2-c and lead him to see a need for sub-dividing the inch into half-inches.

a. Have him mark off half-inch lengths on his ruler (pupil-made ruler). (Distribute to each pupil a $\frac{1}{2}$" long piece of oaktag.)

b. The pupil should be led to understand that the distance from the left edge of the ruler to the line marked 1 is both one inch (1") or two half-inches ($\frac{2}{2}$"). The distance from the left edge of the ruler to the line marked 2 is four half-inches ($\frac{4}{2}$").



1) What is the distance from 1" to 2½"?
2) What is the distance from ½" to 2½"?
3) Name any two numbers which are 2 inches apart; ½" apart.

c. The teacher should review and extend understanding of fractional relationships: (through the use of the ruler)

$2\frac{1}{2} = 2 + \frac{1}{2}$ $\qquad$ $2\frac{1}{2} = \frac{5}{2}$

$2\frac{1}{2} = 1 + 1 + \frac{1}{2}$ $\qquad$ $2\frac{1}{2} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

$2\frac{1}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$ $\qquad$ $2\frac{1}{2} = 3 - \frac{1}{2}$

d. Have pupil express the distance from A to B in terms of the units marked on the ruler. (Approximately 2 inches and ½ inch, or approximately 2½ inches)

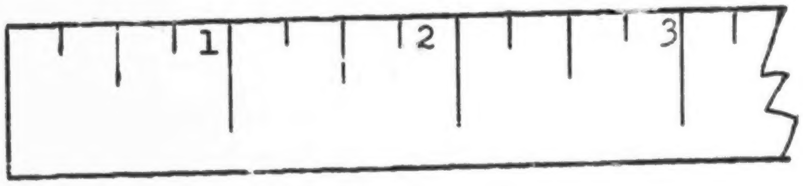
e. Have him use his pupil-made ruler to

1) measure various lengths and
2) draw lines of specific lengths.

4. Have pupil refer to II-A-3-d and lead him to see a need for sub-dividing the half-inch into quarter inches.

a. Have him mark off quarter-inch lengths on his pupil-made ruler. (Distribute to each pupil a ¼" long piece of oaktag.)

b. The pupil should be led to understand that the distance from the left edge of the ruler to the line marked 1 is one inch, or two half-inches, or four quarter-inches. The distance between any two consecutive marks is ¼ inch.



c. The teacher should review and extend pupils understanding of fractional relationships:

1) Express $1\frac{3}{4}$ inches in five different ways.

$1\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$

$1\frac{3}{4} = 1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$

$1\frac{3}{4} = 1 + \frac{1}{2} + \frac{1}{4}$

$1\frac{3}{4} = 1 + \frac{3}{4}$

$1\frac{3}{4} = 2 - \frac{1}{4}$

2) How many $\frac{1}{4}$ inches in one inch? in $\frac{1}{2}$ inch? in $\frac{3}{4}$ inches?

3) $1 = \frac{?}{4}$; $1 = \frac{?}{2}$; $3 = \frac{?}{4}$

4) $\frac{1}{2} = \frac{?}{4}$; $\frac{2}{4} = \frac{?}{2}$; $\frac{6}{4} = \frac{?}{2}$

5) $\frac{3}{4} = 3 \times \frac{?}{4}$; $\frac{6}{4} = 2 \times \frac{?}{4}$; $\frac{6}{4} = 3 \times \frac{?}{4}$

6) $\frac{1}{2}$ of 1 inch = $\frac{1}{2}$ inch; $\frac{1}{4}$ of one inch = $\frac{1}{4}$ inch

7) $\frac{1}{2}$ of $\frac{1}{2}$ inch = $\frac{1}{4}$ inch

$\frac{1}{4}$ of 2 inches = $\frac{2}{4}$ inches or $\frac{1}{2}$ inch

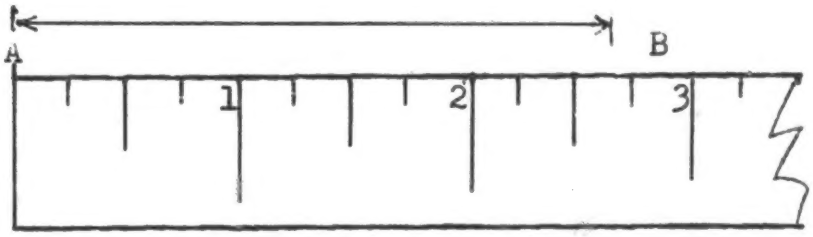
8) $1\frac{1}{2}$ inches = $\frac{?}{2}$; $1\frac{1}{2} = \frac{?}{4}$; $1\frac{3}{4} = \frac{?}{4}$

d. Have pupil express the distance from A to B in terms of the units marked on his ruler.

To the nearest inch, the distance is nearer 3" than 2".

To the nearest $\frac{1}{2}$", the distance is nearer $\frac{5}{2}$" ($2\frac{1}{2}$") than the $\frac{6}{2}$" (3") mark.

To the nearest $\frac{1}{4}$", the distance is nearer $\frac{11}{4}$" ($2\frac{3}{4}$") than the $\frac{10}{4}$" ($2\frac{1}{2}$") mark.



*5. (Optional) Have pupil see need for further sub-division to eighths and sixteenths. Have him use a commercial ruler in a manner similar to that described in II-A-2-4.

*B. Suggested practice (Optional)

1. Use a rule graduated to eighths of an inch. Reading from the left edge of the rule:

   a. What distance is represented by the mark immediately following the 1" mark? the $1\frac{1}{8}''$ mark? the $1\frac{1}{4}''$ mark? the $1\frac{3}{8}''$ mark? the $1\frac{1}{2}''$ mark?

   b. What distance is represented by the mark immediately preceding the 1" mark? the $\frac{3}{8}''$ mark? the $\frac{1}{2}''$ mark?

2. Use your ruler to answer the following questions:

   a. Show on your ruler $1\frac{1}{2}''$ and $1\frac{8}{16}''$. What do you notice?

   b. How much less is $1\frac{3}{4}''$ than $1\frac{11}{16}''$?

   c. If you want to draw a line $\frac{1}{2}''$ long and start at the one-inch mark, where do you stop?

   d. If you want to draw a line $3\frac{3}{8}''$ long and start at the 2" mark, where do you stop?

   e. Which is nearer the 3" mark: $2\frac{7}{8}''$ or $2\frac{15}{16}''$? Explain.

3. a. One inch may be written $\frac{2}{2}''$ or $\frac{4}{4}''$. Represent 1" in several other ways.

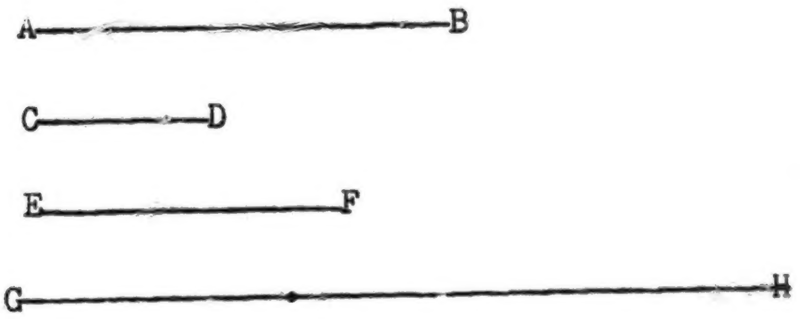
   b. $1\frac{3}{4}$ may be written $\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$, or

   $2-\frac{1}{4}$ or $1+\frac{1}{2}+\frac{1}{4}$. Write $1\frac{3}{4}$ in other ways.

4. Draw lines of such lengths as $2\frac{1}{2}''$; $1\frac{5}{8}''$; and $3\frac{7}{16}''$.

*Optional Topics

5. Estimate the length of each of the following and check by measuring to the nearest $\frac{1}{8}$".



6. Find the distance from

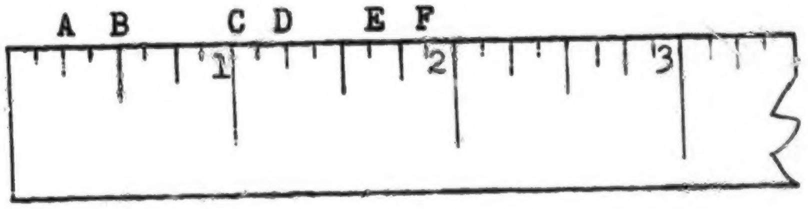
   a. Point A to point C

   b. Point B to point D

   c. Point A to point F

   d. Point C to point D

   e. Point B to point F



III. Approximate Nature of Measurement

A. Suggested procedure

1. Have pupil realize that the result of counting separate objects is usually expressed as an exact number.

   a. Have him count the number of pupils in class, the number of desks, the number of doors, the number of books on the teacher's desk.

   b. He should realize he counted exactly the number of each kind of thing.

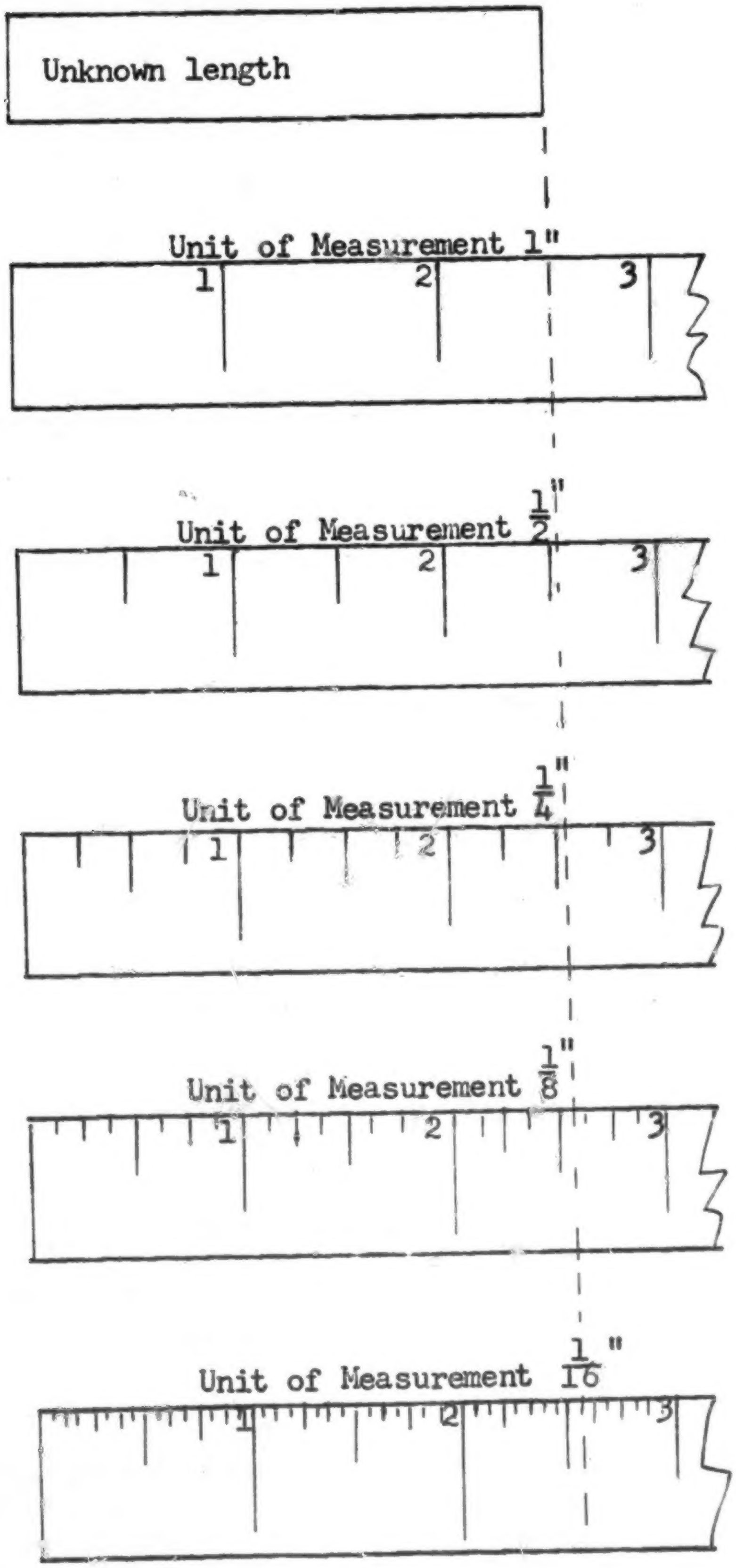
2. Have pupil realize that all measurements are approximate.

   a. Ask pupil how tall he is. He may answer 5' 8 $\frac{1}{2}$" or 68 $\frac{1}{2}$". Have him consider whether he is exactly 5' 8 $\frac{1}{2}$" or 68 $\frac{1}{2}$". Is he perhaps 5' 8 $\frac{3}{8}$", or perhaps 5' 8 $\frac{5}{8}$"?

   b. Have pupil realize that he could get closer to his exact height by using a ruler divided into sixteenths or thirty-seconds. He should begin to realize that it is always possible to continue to get a smaller unit of measurement. Therefore, no measurement will give him his exact height.

Note: The vertical marks on a ruler are themselves several thousandths of an inch thick.

c. Have pupil use different units of measurement in measuring lengths to develop an understanding of approximate measures. Have him measure an unknown length to the nearest 1"; $\frac{1}{2}$"; $\frac{1}{4}$"; $\frac{1}{8}$"; $\frac{1}{16}$".


Unknown length
Unit of Measurement 1"
1
2
3
Unit of Measurement $\frac{1}{2}$"
1
2
3
Unit of Measurement $\frac{1}{4}$"
1
2
3
Unit of Measurement $\frac{1}{8}$"
1
2
3
Unit of Measurement $\frac{1}{16}$"
1
2
3

Measurement is between 2" and 3". It is 2" to the nearest inch.

Measurement is $2\frac{1}{2}$" to nearest $\frac{1}{2}$".

Measurement is $2\frac{2}{4}$" to nearest $\frac{1}{4}$".

Measurement is $2\frac{5}{8}$" to nearest $\frac{1}{8}$".

Measurement is $2\frac{9}{16}$" to the nearest $\frac{1}{16}$".

Have the pupil conclude that the smaller the unit used, the closer the measurement is to the exact length of that which is being measured. The result of a measurement is always approximate. Each measurement has been rounded off.

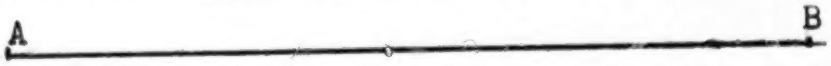
B. Suggested practice

1. Examine the labels on a spool of thread, ball of twine, package of loose-leaf paper, and box of tea bags. Determine whether the number describing the contents in each case is exact or approximate. Give a reason for your answer.

2. State in which instances the numbers are exact:

   a. 5 books
   b. 7 books
   c. 3 pupils
   d. 4 desks
   e. $8\frac{1}{4}$"
   f. 3 lbs.
   g. 6 tons
   h. $1\frac{1}{2}$ miles
   i. $3.22

3. Measure the line segment AB to the nearest inch, $\frac{1}{2}$", $\frac{1}{4}$", $\frac{1}{8}$", $\frac{1}{16}$".



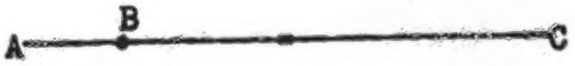
4. Draw lines of the following lengths:

   4"  $2\frac{1}{2}$"  $3\frac{3}{4}$"  $1\frac{7}{8}$"  $\frac{15}{16}$"

5. a. Measure AB_____

   b. Measure BC_____

   c. Measure AC_____



   Note: Have pupil realize that if he adds the two smaller lengths, he will find the total length.

IV. Precision

A. Suggested Procedure

1. Have pupil realize need for precise measurement.

   a. Have him discuss situation in which precise measurement would be involved:

      1) Eye chart - placed to nearest 1"
      2) Shelving - trimming shelves to nearest $\frac{1}{2}$"
      3) Hems - length of skirt from the ground to nearest $\frac{1}{4}$"
      4) Window panes - to nearest $\frac{1}{8}$"

b. Have him discuss mass production in connection with precise measurement.

c. Have him try to find out the precision used in various occupations.

| Worker | Measuring Instrument | Degree of Precision |
|---|---|---|
| Printer | Pica Gauge | $\frac{1}{6}$" |
| Woodworker | Two-foot Rule | $\frac{1}{8}$" |
| Dressmaker | Tape | $\frac{1}{4}$" |
| Carpenter | Steel Square | $\frac{1}{16}$" |
| Auto Mechanic | Feeler Gauge | .001" |
| Toolmaker | Micrometer Caliper | .001"or .0001" |

Conclusion: The precision with which we measure depends upon the work to be done.

2. Have pupil realize that the smaller the unit of measure, the more precise the measurement.

   a. Have pupil measure an unknown length as shown in III-A-2-c.

   b. Have pupil conclude that the smaller the unit of measure, the more precise the measurement. The more precise the measurement, the closer will be the measured length to the exact length.

   c. When measuring to a certain unit, it is necessary to round the measurement to that unit.

B. Suggested practice

1. Is $2\frac{7}{8}$" or $2\frac{15}{16}$" nearer to 3"?

2. Name the unit of measurement used in each of the following:

   a. 7"
   b. 3'
   c. 4'
   d. $3\frac{1}{2}$"
   e. $5\frac{1}{4}$"
   f. 4 yards
   g. $1\frac{7}{8}$"
   h. $3\frac{5}{16}$"
   i. $2\frac{4}{8}$"

3. Which measurement is most precise?

   $1\frac{1}{2}$" $1\frac{2}{4}$" $1\frac{4}{8}$" $1\frac{8}{16}$"

4. Which measurement is more precise in each of the following pairs? Explain.

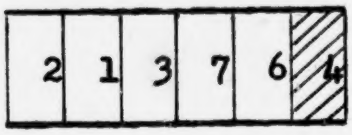
| | |
|---|---|
| a. $3\frac{7''}{8}$, $12''$ | d. $2\frac{1''}{4}$, $3\frac{3''}{8}$ |
| b. $21\frac{1''}{4}$, $3''$ | e. $4\frac{0''}{4}$, $3\frac{1''}{2}$ |
| c. $4\frac{1''}{2}$, $4\frac{4''}{8}$ | f. $4\frac{5''}{8}$, $4\frac{10''}{16}$ |

## V. Measurements and Decimal Fractions

### A. Suggested procedure

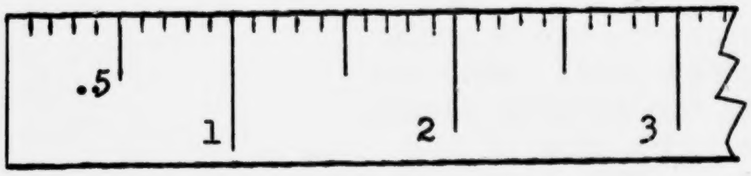
1. Have pupil realize there are instruments which can be used to make measurements to .1 inch, 01 inch, 001 inch. (There are instruments for measurement which are even more precise.)

2. Have pupil consider a car odometer which measures to tenths of a mile.



Have pupil learn to read these numbers. Review meaning of tenths. Have him count by tenths and realize 10 tenths is 1.

3. Have pupil learn to read a ruler divided into tenths of an inch.



Have him consider such questions as:

a. How many tenths in one inch?
b. How many tenths in two inches?
c. How many tenths in a half inch?
d. Is .4 inch more or less than a half inch?
e. How else may ten tenths be expressed?

4. Show pupil a micrometer caliper and tell him it measures to nearest .001". Use the micrometer to measure the thickness of a piece of paper or a hair. The pupil should begin to realize that even the marks on his ruler can be measured. Have him consider the need for parts to be interchangeable in mass production. Therefore, the parts must be carefully made to within certain limits. The limit is frequently stated in .001".

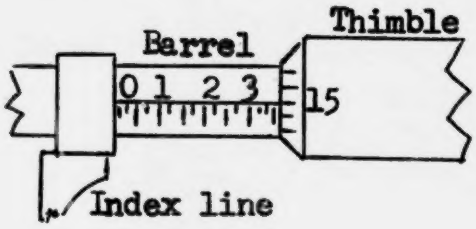
Note: The study of micrometer caliper is optional. It is included to:

a. Show instruments are available and in use in industry and laboratories to measure length precisely.

b. Reinforce skills in addition and multiplication of decimals.

c. Develop relation of common fraction and decimal fraction.

*5. Teach pupil how to read the micrometer caliper scales.



a. Each space on the barrel represents .025". Count the number of whole spaces on the barrel and multiply by .025.

$$\begin{array}{r} .025 \\ \times \underline{\;15} \\ 125 \\ \underline{25\;} \\ .375 \end{array} \quad .375$$

b. Each space on the thimble represents .001. Read the line on the thimble closest to the index line of the barrel.. .015

c. Add the results. This is the size of the object being measured.

$$\begin{array}{r} .375 \\ \underline{.015} \\ .390 \end{array}$$

B. Suggested practice

**Note**: Pupil must be taught to read the decimal point as, "and" i.e., 24.6 should be read as, "twenty-four and six tenths." Emphasize that the number names for all numbers on the right side of the decimal point end in "th" or "ths".

1. Have pupil read

a. 24.6
b. 357.9
c. 4.0
d. 6.08
e. 37.91
f. 486.01
g. 7.00
h. 13.001
i. 1.000

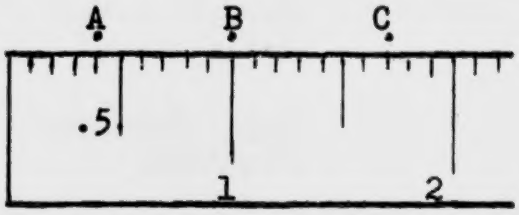
2. How many tenths in: (as in V-A-3)

a. 3"
b. $\frac{1}{2}$"
c. 1 $\frac{1}{2}$"
d. 4"
e. 1.2"
f. .30"

3. Distance A to B is _____

Distance B to C is _____

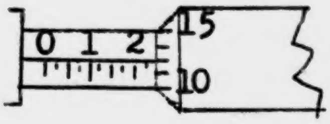
Distance A to C is _____



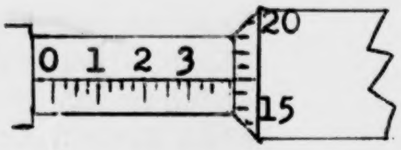
4. a. $\begin{array}{r} .025 \\ \text{x}\ \underline{19} \end{array}$ b. $\begin{array}{r} .025 \\ \text{x}\ \underline{18} \end{array}$ c. $\begin{array}{r} .025 \\ \text{x}\ \underline{17} \end{array}$

5. What is the reading?

a.



b.



## III.

## GEOMETRY UNIT

I. Development of the Concept of Angle

A. Suggested procedure

1. Pose the following situations

a. Juan and Jack agree to meet at the Community Center. Juan is new to the neighborhood. What information must Jack give him so that he can find the Center? (street and number) (accept either answer or both)

b. They agree to meet Juan's friend, Rafael, at Joe's corner candy store for sodas when the center is closed. What information must Jack give Rafael to enable him to locate Joe's store? (names of two streets which intersect)

Note: Have pupil see that two elements of reference are required.

c. A sailor on board a tanker falls ill and needs a doctor. The ship's captain calls the Coast Guard via radio and asks for help. The Coast Guard wishes to dispatch a helicopter. How does the Coast Guard locate the ship?

Discussion:

Because streets and numbers do not exist on the ocean, positions of ships are reported by stating latitude and longitude. Another way would be to state the distance and direction of the ship from a Coast Guard Station. In either case, the Coast Guard must determine the direction in which the helicopter must fly.

2. Angle and direction

a. Have pupil recall the four major compass points (directions).

Note: It is suggested that a map of New York State be used for this section because each pupil can obtain a New York State road map at a local gas station. The teacher may substitute any map and any cities that would serve the same purpose with corresponding changes in questions, e.g., map of the United States.

1) Pose the following questions:

If a pilot had to fly from New York City to Schenectady, in what direction should he head? (North)

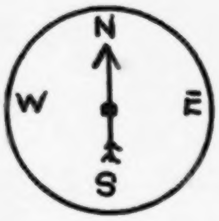
If a pilot had to fly from Utica to Niagara Falls, in what direction should he head? (West)

If a pilot had to fly from Buffalo to Auburn, in what direction should he head? (East)

If a pilot had to fly from Schenectady to New York City, in what direction should he head? (South)

2) Have pupil observe that from New York City to Schenectady, the heading is North, and that from Schenectady to New York City the heading is South. Although the same two cities are involved, the direction is different. Explain. (different point of reference)

3) The direction compass

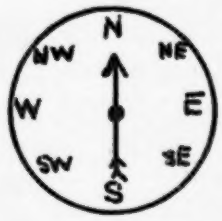


What instrument is used for determining direction? (direction compass)

What do the four letters, N, E, W, S, on the face of the compass represent?

b. Other (secondary) directions

1) As in 2-a-1) and 2), develop the directions NE, SE, SW, NW.



Binghamton to Amsterdam (NE)
N.Y.C. to Auburn (NW)
Binghamton to Jersey City (SE)
Amsterdam to Binghamton (SW)

Have pupils determine that the directions NE, SE, SW, NW lie midway between each of the four major direction points.

2) Have pupil realize that a direction midway between N and NE might be called NNE but that naming all intermediate directions would be very cumbersome. Therefore, another method of determining direction should be sought.

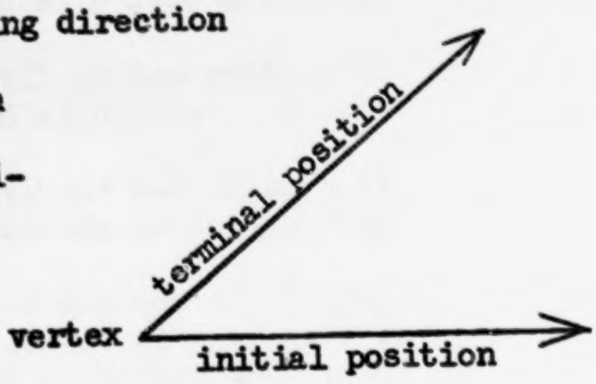
c. The concept of the angle in determining direction

1) Have pupil review the concept of angle.

a) the position of a line
b) vertex as point of reference about which line turns
c) terminal position of line

3. The use of the angle in determining direction

a. Review the concept of rotation (Grade 7 Mathematics, p. 155, #4) stressing the initial position of the line, the point about which the line turns (vertex) and the terminal position of the line.

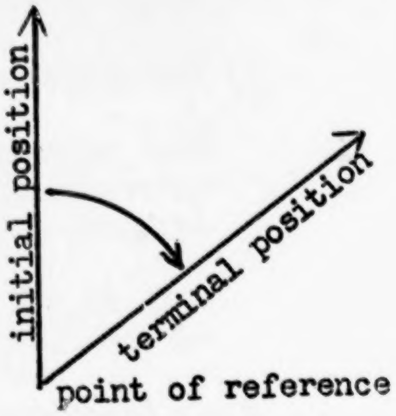

terminal position
vertex
initial position

b. Have pupil relate this concept to direction.

1) The vertex is the center of the compass (point of reference).

2) The initial position of the line is the line drawn from the vertex (point of reference) due north.

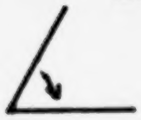
3) The terminal position of the line is called the heading or bearing from the reference point to the destination measured clockwise from the N. (initial position of the line)


initial position
terminal position
point of reference

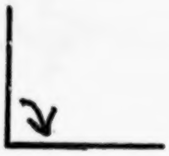
c. Extension of the concept of the angle (reflex angle)

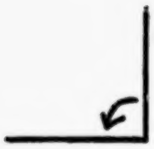
1) Review angle classifications

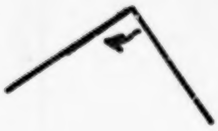
a) Acute angle: greater than zero (0°) and less than 90 degrees (90°).



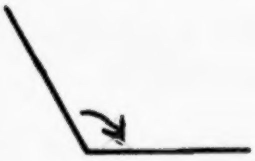
b) Right angle: 90 degrees (90°)

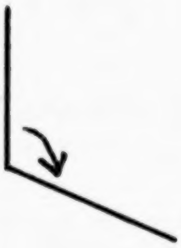


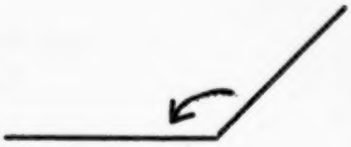




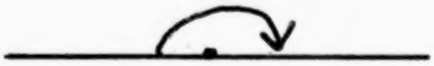
c) Obtuse angle: greater than 90 (90°) and less than 180 degrees (180°)

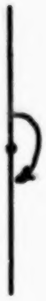






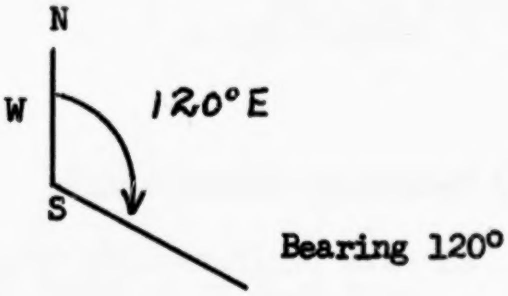
d) Straight angle: 180 degrees (180°)





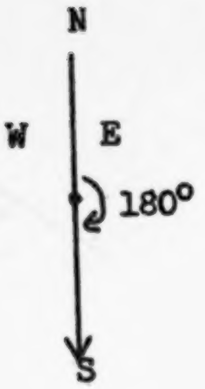
2) Reflex angle (introduction)

a) Have pupil recall -see above 2-c-2)-c) - that in measuring angles in direction, the bearing is measured clockwise from initial position. (North)



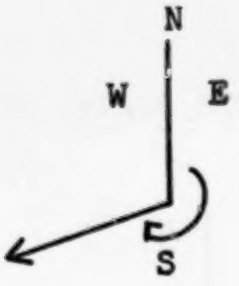
b) Have pupil realize that a bearing west of south would result in an angle greater than a straight ∠ (180°).

1)

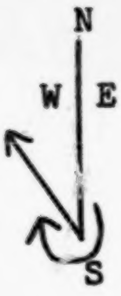


Bearing South
(180°)

2)



3)



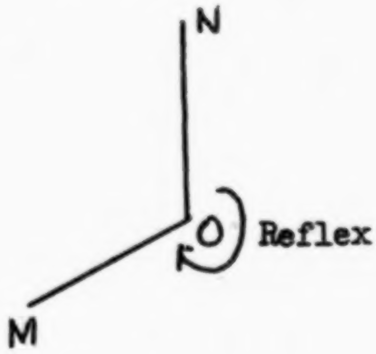
c) Have pupil observe that this type of angle - figures 2) and 3) - is called a reflex angle.

d) Indicate to pupil that a reflex angle is an angle greater than 180° but less than 360°.

e) Have pupil realize that he must specifically identify a reflex angle, i.e., reflex ∠ NOM.

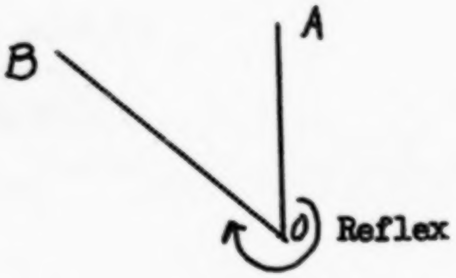
∠NOM refers to obtuse ∠NOM ;

reflex ∠NOM refers to the larger of the two angles.



∠AOB refers to acute ∠AOB;

reflex ∠AOB refers to the larger angle.

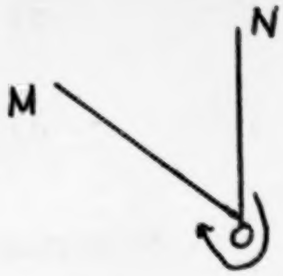


3) Measuring reflex angles

a) Have pupil realize that since his protractor measures only from 0° to 180°, there is a need for a method to measure reflex angles.

b) Elicit from pupils possible methods for measuring reflex angles.

c) Have pupil realize that if he measures the smaller angle and subtracts from 360°, he will have measured the reflex angle indirectly (subtraction method).



Pupil measures acute ∠NOM

(e.g., ∠NOM = 50°)

He then subtracts 50° from

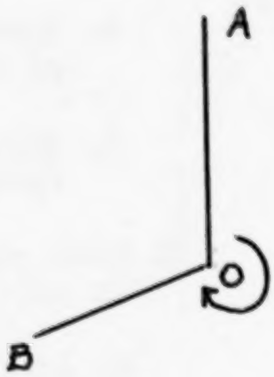
360°.

$$\begin{array}{r}360^\circ\\ -\ 50^\circ\\ \hline 310^\circ\end{array}$$

Therefore, reflex ∠NOM = 310°

Have pupil measure obtuse ∠AOB.

Find reflex ∠AOB.



4. The use of the protractor

a. Review the use of the protractor (Grade 7 Mathematics, p. 157, item h).

1) The protractor must be lined up with the initial line (either line may be used as the initial line). The center must be placed at the vertex.

2) Selection of the proper scale

a) When the angle is less than 90°, select the smaller number (acute angle).

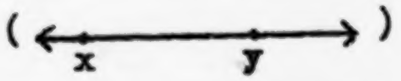
b) When the angle is more than 90°, select the larger number (obtuse angle).

b. Practice

1) Prepare mimeographed sheet with ∠s for measuring.

2) Drawing angles

a) Draw five lines and label a point x and a point y on each.



b) At x, in each line, draw an angle of 30°, 60°, 120°, 190°, 300°.

c) At y, in each line, draw an angle of 45°, 75°, 135°, 255°, 305°.

d) Have pupil practice drawing and measuring angles with initial lines xy in vertical position.

5. Measuring directions in degrees

   a. Have pupil attempt to describe the direction in which a plane would fly from Buffalo to Rochester. (Can not be described accurately in terms of N, NNE, NE, ENE, or E)

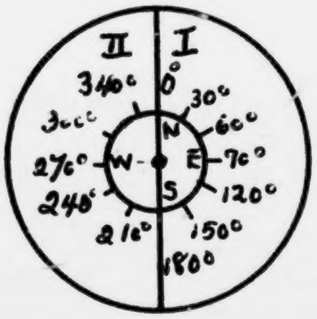
   b. Question: What might be a better method of directing the pilot of the plane? (Give pilot his course in degrees.)

   c. Elicit from pupil a familiar method for measuring angles (protractor).

6. Have pupil recognize the fact that a compass is a protractor in the form of a complete circle.

   a. If a protractor (half circle) contains 180°, how many degrees are there in the full circle? (360°)

   b. Have pupil construct a simple compass face by using his own protractor. (See diagram and note position I and position II.)



7. Demonstrate to pupil that using this instrument, any direction can be described in terms of number of degrees from North (clockwise). For example, the direction from Buffalo (vertex) to Rochester (destination) is approximately 75°. This direction measured clockwise from the North is called bearing. Therefore, the bearing from Buffalo to Rochester is 75° (approximately). Reflex angles can be measured directly.

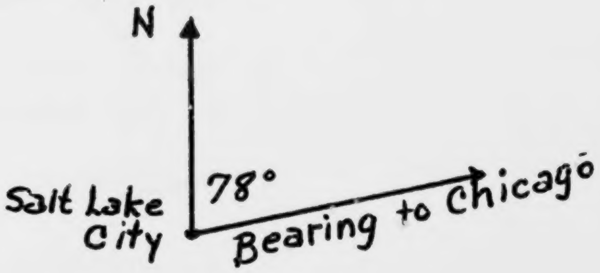
Note: Vertex - Buffalo

Initial position of line - line from Buffalo (vertex) running due north (0°)

Terminal position of line - line from Buffalo (vertex) to Rochester. The initial position of the line is always the line from the vertex running due north.

B. Suggested practice

1. In the following examples, you are asked to make a diagram of the line of flight. Draw the North-South line (initial position of line). Select a point on this line to represent the first city. Indicate the line of flight (bearing) between the cities.



| | Bearing |
|---|---|
| a. Salt Lake City, Utah to Chicago, Illinois | |
| b. St. Louis, Missouri to Atlanta, Georgia | 132° |
| c. Detroit, Michigan to New York, N. Y. | 100° |
| d. Chicago, Illinois to Detroit, Michigan | 80° |
| e. San Francisco, California to Salt Lake City, Utah | 68° |
| f. Denver, Colorado to Phoenix, Arizona | 223° |
| g. New York, N. Y. to Buffalo, N. Y. | 300° |
| h. New York, N. Y. to Syracuse, N. Y. | 310° |

II. Applications (other than direction)

Note: Many aspects of the use of the angle can be found in industry and construction. Some are listed here. Bring to the attention of the class additional specific instances.

A. Revolutions per minute and the angle

1. Pose the following questions:

a. How many degrees of rotation will a wheel make in one complete turn? (360°)

b. How many degrees in 2 complete turns? (720°)

c. How many degrees in 5 complete turns? (1800°)

d. How many degrees in 10 complete turns? (3600°)

2. Discuss with pupil that in industrial machines, automobiles, etc., wheels make as many as 1,000, 4,000, 10,000 complete turns in one minute.

3. Have pupil realize that expressing these rotations in terms of the number of degrees of rotation in one minute would result in very large numbers. Have him see that some other notation for angle measurement is desirable.

4. Elicit from pupil that 360° turns in one minute is expressed as the number of revolutions per minute (rpm)

5. Practice

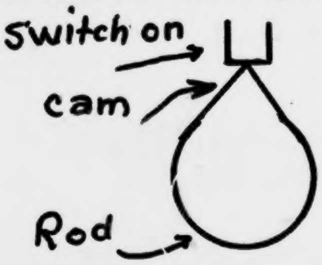
   a. Indicate the number of degrees of rotation in one minute for each of the following:

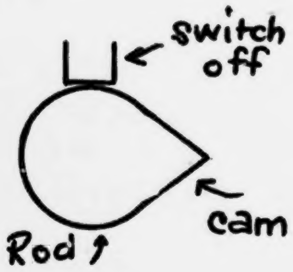
      2 rpm.
      10 rpm.
      100 rpm.
      150 rpm.
      1,000 rpm.
      2,400 rpm.

   b. In the above, indicate the number of degrees of rotation in two minutes for each.

B. Cam shafts and the angle

1. Discuss the nature of the cam shaft with pupil. A cam shaft is a rod with a bump on it. As the rod rotates, the "bump" (cam) will turn a switch on and then off.

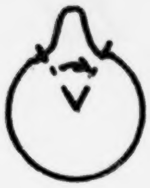




2. Pose question:

   Have you ever wondered why the alarm of a clock will continue to ring for such a long time after it starts?

   a. Discuss with pupil the fact that the designer can control the length of the alarm by determining where the "bump" (cam) starts and ends. The number of degrees in the central angle covered by the cam on the rod (see diagram) determines this.



3. Suggested practice

   Draw a diagram of the following cam shafts:

   a. a cam of $20^{\circ}$
   b. a cam of $30^{\circ}$
   c. a cam of $35^{\circ}$
   d. a cam of $50^{\circ}$

C. Shop work and the angle

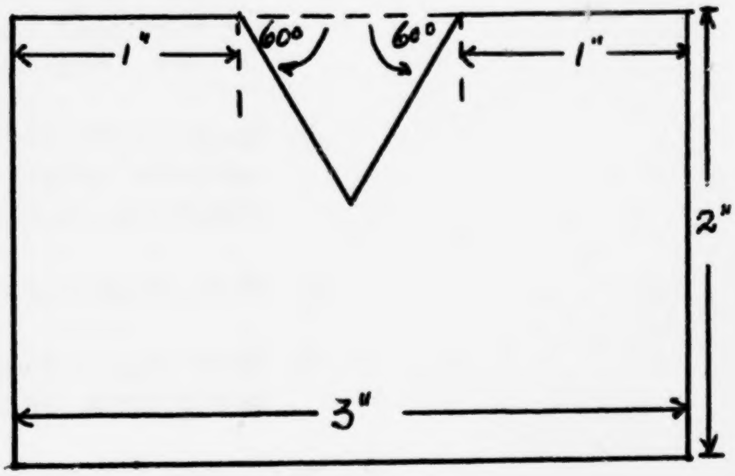
1. Discuss with pupil the possible use of the angle in shop work.

   a. carpenter's square for right angles
   b. table legs set in at an angle
   c. the V block
   d. the angle of the cutting edge of a drill bit

2. Suggested practice

   a. Making a V block

      A block is 3" wide and 2" high. A $60^{\circ}$ V cut is to be made at the center. Draw a diagram to demonstrate this operation.

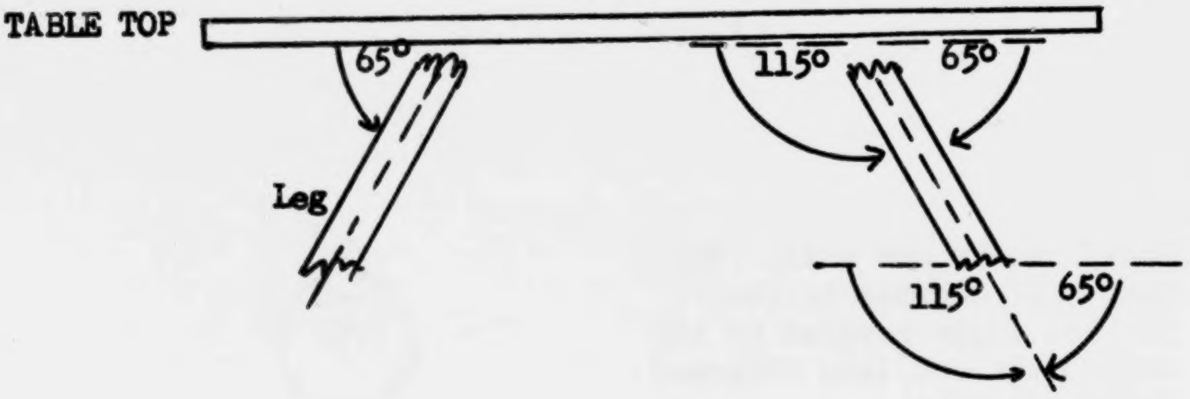


      1) Have pupil draw a rectangle that is 3" long and 2" high.

      2) Have pupil mark off points 1" from each edge of block (see diagram)

      3) Measure an angle of $60^{\circ}$ at each of these points.

   b. Legs are to be attached to a table so that they make an angle of 65o with the top. Indicate by a diagram how the legs must be cut.



## III. Concept of Area

Recommendation to Teacher: Since pupils have been introduced to area and volume in previous grades, a pretest is suggested to eliminate unnecessary lessons and to permit the teacher to move to the more advanced work in this section.

Note: It is recommended that the pupil draw diagrams wherever possible.

### A. Suggested procedure

1. Have pupil review measuring parts of lines (line segments).
   a. Have pupil answer the following questions after measuring:
      1) How long is this sheet of notebook paper? (demonstrate)
      2) How wide is your desk?
      3) How high is the board?
      4) How long is the board?
      5) How long is the room?

   b. Have pupil understand that when he measures, he compares the unknown length with a known length called the unit. His ruler has units marked on it.

   c. Have pupil realize that he measured parts of lines.

   d. Have pupil see that whenever he is using the ruler as an instrument of measure, he is measuring one dimension - length.

2. Help pupil develop greater insight into the meaning of surface.
   a. Ask pupil to find the size of his desk.
      1) Permit pupil to make any measure he elects.

      2) Pupils will realize that measuring length alone or width alone will not help them decide the size of their desks.

3) Have pupil conclude that two dimensions, (length and width), must be stated to express the size of the surface (desk).

4) Have pupil express the size of various surfaces in terms of 2 dimensions, e.g., pupils' desks 15" x 12" (read 15" by 12". Board 12" x 3½'. Room 28' x 22'.

b. Concept of surface measure

1) Have pupil answer the following question:

Mr. Jones plans to paint his living room. How does he determine the amount of paint to buy? (by determining the amount of surface to be covered)

Note: Each gallon can of paint indicates average amount of surface the paint will cover (usually 400 sq. ft.).

2) The pupil should realize that the measurement of surface has many practical applications.

Have pupil give instances where it is necessary to know the amount of surface, e.g., tiling a bathroom, papering a wall, silvering a mirror, seeding a lawn, carpeting a floor, cementing a walk, etc.

3) Have pupil realize that it is convenient to have standard units with which to measure surface. With these units accepted, people can purchase correct amounts of tile, wallpaper, etc.

a) Have pupil attempt to measure a surface with a ruler.
b) Lead pupil to realize that he cannot measure a surface with an instrument that measures lines because he cannot cover the surface.

3. Developing units for measuring surface

a. Have pupil recall units which he has observed are used to cover surfaces:

bathroom tiling - floor and walls
asphalt tiles
ceramic tiles (table tops)
sectioned wall mirror

b. Have him describe the shape of these units. (square, less frequently hexagon)

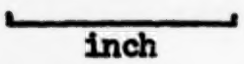
c. Have him consider why the manufacturer selects the square as the unit for covering. Have him try to cover a surface using other shapes, e.g., a circle or a triangle to verify this conclusion.

d. Have pupil conclude that the most convenient unit for measuring a surface is a square.

e. Have pupil recall the standard units of linear measure (inch, foot, yard) and have him consider the convenience of analogous units for measuring surface.

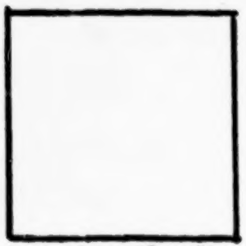
1)

Linear Measure



inch

Surface Measure



square inch

2) Have pupils illustrate on blackboard:

One linear foot; one square foot

3) Have pupils illustrate on blackboard:

One linear yard; one square yard

Note: Show the square yard divided into square feet.

### B. Practice

Tell which of the following are instances of linear measure or of surface measure:

1. the amount of thread on a spool (L)
2. a boy's height (L)
3. a plank, a slat, or a wire fence to be whitewashed (S)
4. a floor to be sanded (S)
5. a fringe to be put on a bedspread (L)
6. a rug to be cleaned (by square foot) (S)
7. a framed painting on a wall (S)
8. Have pupil make up more examples of this type.

C. Specific outcomes

1. To measure a length, a unit length is used.
2. To measure surface, a unit of surface must be used.
3. The usual unit of surface is a square.
4. Some units of surface are: square inch, square foot, square yard, square mile.

## IV. Measurement of a Rectangular Surface

A. Suggested procedure

1. Have pupil measure directly.

    a. Have pupil use the square inch to measure directly.

        1) Have pupil measure a piece of paper 6" long and 3" wide.

        Note: Provide each pupil with inch squares. (May be ordered from kindergarten list or made by pupils.)

        2) Have pupil find surface by counting or multiplying number of squares in one row by number of rows. (18 sq. in.)

    b. Have pupil use the square foot to measure directly.

        1) Have pupil measure a section of a board 4 ft. by 3 ft, by applying a sq. ft. (oaktag or paper) and marking off each square foot.

        2) Have him count to see that this section of board measures 12 sq. ft.

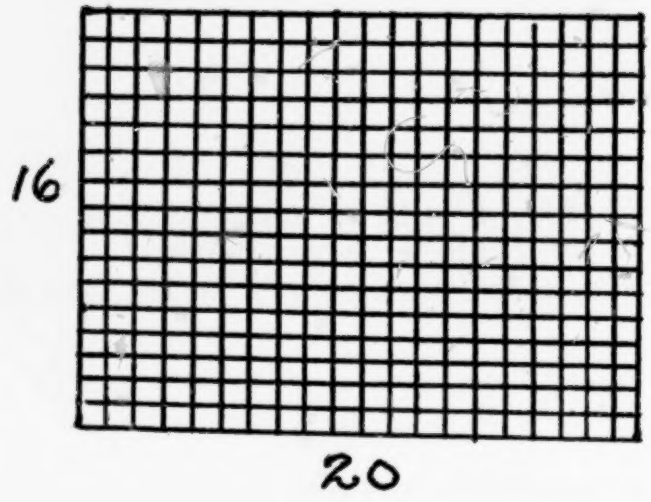
2. Have pupil use squared paper to measure.

    a. Pose problem

    Mr. Jones plans to install tile ceiling in his living room. The ceiling is a rectangle 16' x 20'. It is necessary for him to know exactly how many tiles he needs for the job as the storekeeper will not take back any extra tiles. Each tile is a square 1' x 1'. How many tiles should he buy?

        1) Have pupil diagram the ceiling on squared paper.

        (Scale: 1 square = 1 tile)



2) Have pupil suggest ways of determining the number of tiles needed.

   a) counting the squares (320)
   b) multiplying the number of tiles in one row (20) by the number of rows (16); 20 x 16 = 320.

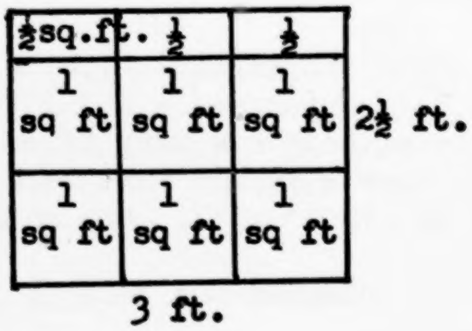
3) Have pupil recognize that he has been measuring the surface of the ceiling with a unit of 1 square foot. (Each tile is 1' by 1')

4) Have pupil realize that the 320 tiles represent the surface of the ceiling which measures 320 sq. ft.

5) Have pupil learn that the number of square feet (32) is called the area of the ceiling.

b. Have pupil find the area of a rectangular table 3 feet long and 2½ feet wide.

   1) Have him draw a diagram on square paper (1 foot is represented by the length of 1 square).



| | | | |
|---|---|---|---|
| ½sq.ft. | ½ | ½ | |
| 1 sq ft | 1 sq ft | 1 sq ft | 2½ ft. |
| 1 sq ft | 1 sq ft | 1 sq ft | |
| 3 ft. | | | |

   2) Have pupil find the number of square feet (squares) in the surface of the table

a) by counting and adding (6 whole squares + 3 half squares; total = 7½ squares),

b) by multiplying the number of squares in a row (3) by the number of rows (2½)

$3 \times 2\frac{1}{2} = 3 \times 2 + 3 \times \frac{1}{2} = 6 + 1\frac{1}{2} = 7\frac{1}{2}$

3) Have pupil conclude that an area can be expressed as a whole number of squares (7) and a part of a square (½).

c. Have pupil find areas of other rectangles by using graph paper and counting, e.g., 2½' by 3'; 4½" x 2½".

d. Lead pupil to see that diagramming and counting squares is a cumbersome method of finding the area and that a shorter method is desirable. (Have him attempt to find by diagramming and counting squares, the area of a piece of metal $2\frac{11}{16}$" by $1\frac{1}{8}$" to illustrate the point.)

3. Have pupil learn to find the area of a rectangular figure by computation (indirectly).

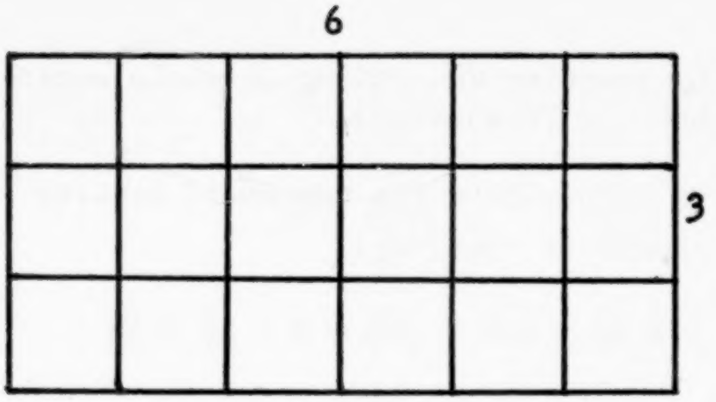
   a. Supply pupil with paper having dimensions which are integral, e. g., 6" x 9", or 3" x 6", or rexographed paper of rectangles with integral dimensions.

   b. Elicit from pupil that since he has no squares to work with, he must find some other way to determine the total number of squares in the surface of the paper.

   c. Have pupil realize that since the square inch is related to the linear inch, he can place his ruler along the length (base) of the rectangle to determine the number of rows of squares, and by placing his ruler along the width (height) of the rectangle, he can determine the number of squares in each row.

   d. Have him find the area of the 3" x 6" paper (18 sq. in.).

   e. Have pupil generalize that you can compute the number of square units in a rectangle by multiplying the number of units in the length (base) by the number of units in the width (height).



(To find the area, multiply the numbers 6 and 3 and attach the label, "square inches," since we are not multiplying inches by inches, but the number of square inches in a row by the number of rows.)

f. Have pupil state this rule as a formula. Verbally: Area equals the base multiplied by the height.

Algebraically: $A = b \times h$

or

$A = bh$

g. Sample solution

A classroom is 35' x 31'. How many square feet are there in its area?

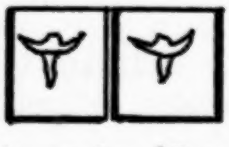
Estimation: about 1200 square feet (40 x 30)

$A = bh$
$A = 35 \times 31$
$A = 1085$

The area of the room is 1085 square feet.

B. Suggested practice

1. Ida has a tray made up of two ceramic tiles. Have pupil diagram this. What is the area of the tray in terms of the tiles?



2. Each tile in Ida's tray is 6" x 6". Draw a diagram. How long is the tray? How wide is the tray? How many square inches of tile surface has the tray?

3. Express the length of Ida's tray in feet. Express its width in feet. What is its area in terms of square feet?

4. Show that the surface measure in #3 equals the surface measure in #2. (There are 144 square inches in 1 square foot.)

5. Mrs. Martin wishes to have a rectangular mirror 30" x 18" resilvered. The cost is 2¢ per square inch. What will be the cost of resilvering the mirror?

6. Mrs. Johnson bought a rug for her husband's den. It measured 12 feet by 15 feet. At $11 a square yard, how much did it cost?

Suggested procedure:

How many yards long was the rug?
How many yards wide was the rug?
How many square yards of rug did Mrs.Johnson buy?
At $11 a square yard, how much did the rug cost?

C. Specific outcomes

1. The number of square units in a surface is called its area.
2. The area of a rectangle can be found by counting the number of squares in its surface.
3. A shorter way of finding the area of a rectangle is by computing (multiplying the number of square units in one row by the number of rows).
4. The formula A = lw or A = bh gives the rule for computing the area of a rectangle.

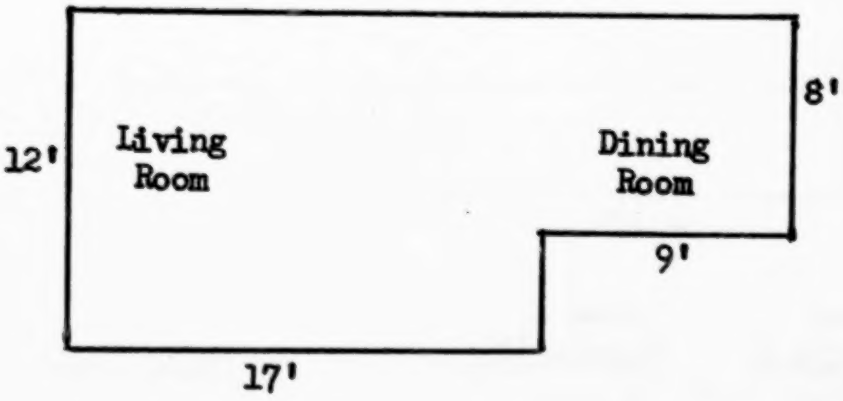
## V. Finding Areas of Irregular Surfaces Which Can Be Resolved Into Rectangles

A. Suggested procedure

Pose problem:

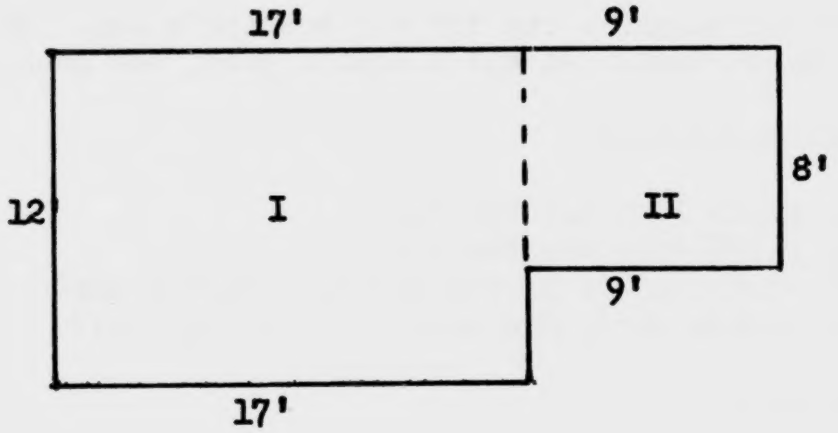
Mrs. Parris plans to purchase wall-to-wall carpeting for her living room and dining room with the following shape and dimensions.

Figure A



1. Have pupils suggest ways of finding this area.
   Have pupils suggest finding the area of the figure by dividing it into rectangles and adding their area.

   Figure B

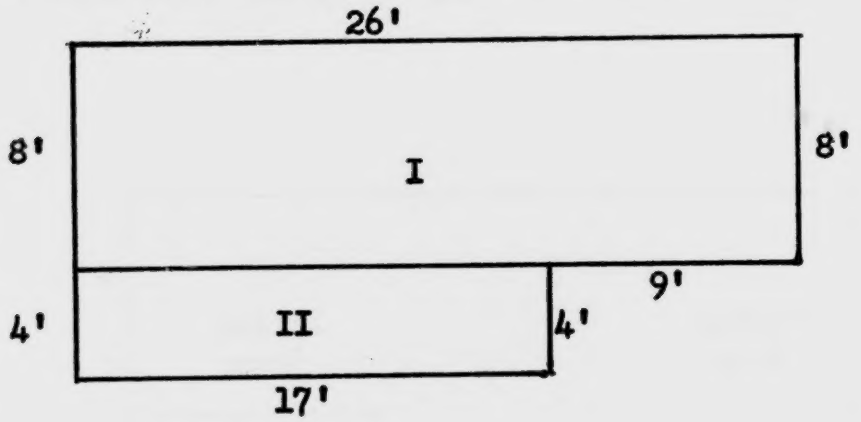


2. Have pupils determine how to find area I and area II.

3. Solution

| Area<br>Figure I | Area<br>Figure II | |
|---|---|---|
| | | Total area = area of Fig. I + area of Fig. II |
| A = bh | A = bh | |
| A = 17x12 | A = 9x8 | Total area = 204 + 72 |
| A = 204 sq.ft. | A = 72 sq. ft. | Total area = 276 sq. ft. |

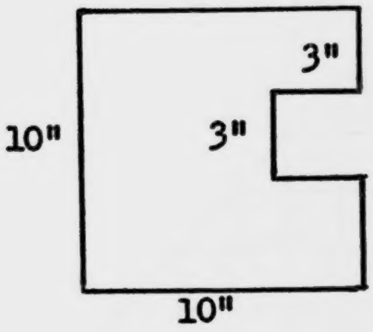
4. Pupils may suggest an alternate solution.

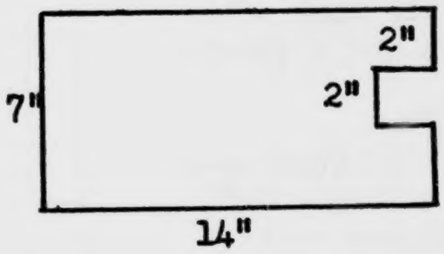


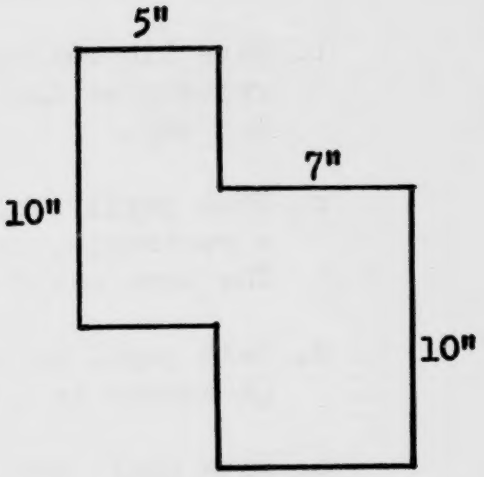
| Area<br>Figure I | Area<br>Figure II | |
|---|---|---|
| | | Total area = area of Fig. I + area of Fig. II |
| A = bh | A = bh | |
| A = 26x8 | A = 17x4 | Total area = 208 + 68 |
| A = 208 sq.ft. | A = 68 sq. ft. | Total area = 276 sq. ft. |

B. Suggested practice

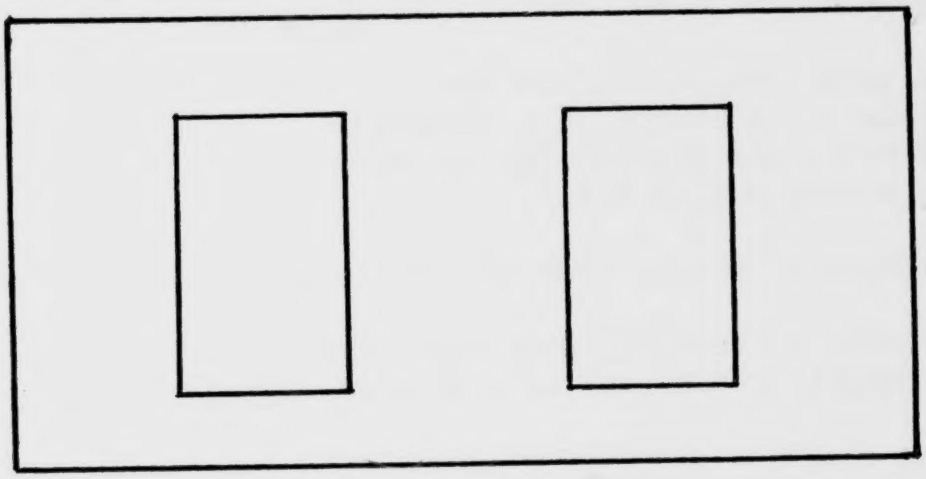
1. Find the area of each of these figures.







2. This drawing represents a wall of a room which is to be painted. The room is 16 ft. wide and 8 ft. high. The wall has two windows, each 3 ft. wide and 5 ft. high.



a. Find the area of the wall including windows.
b. Find the area of the wall minus the windows.

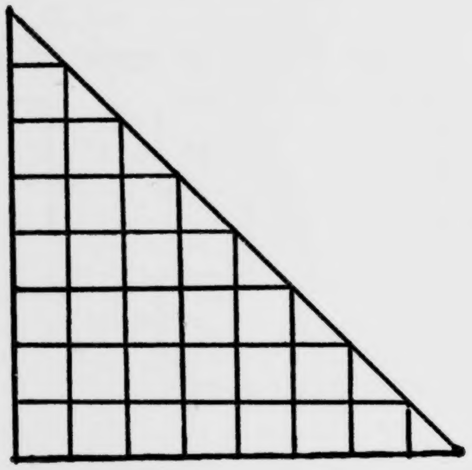
3. Find other irregular figures of this type and compute the area. (Sunday paper - Real Estate Section)

## VI. Developing the Rule and the Formula for the Area of a Triangle

A. Suggested procedure

1. Pose problem:

What is the area of a piece of ground in the shape of a right isosceles triangle with base 8 feet and height 8 feet? (See figure. Draw on squared paper.)



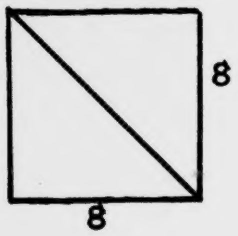
a. Have pupil suggest ways of finding area. (counting)

b. Have him realize from his work with rectangles that this is not a practical way.

c. Have pupil realize that if this were a rectangle, he could multiply 8x8. The area would be 64 square units.

d. Help pupil build up a rectangle. (A square is a rectangle.)

e. Have pupil see that the area of the right triangle is ½ the area of the rectangle. (Pupil can fold such a rectangle along the diagonal and see that two equal triangles are formed.) The area of the triangle is ½ of 64 or 32 sq. in.

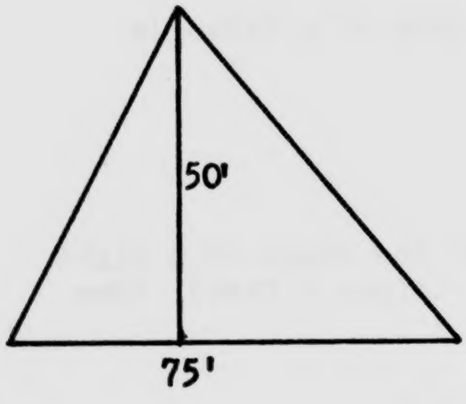


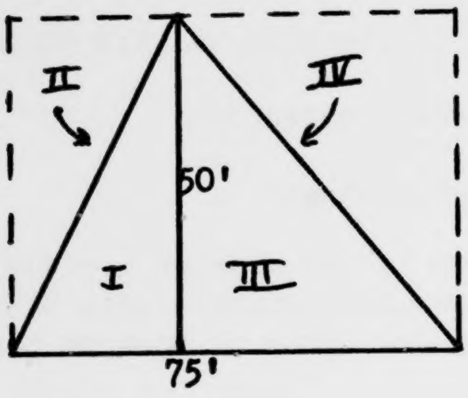
f. Have pupil see that since the formula for the area of a rectangle is $A = bh$, the formula for the area of a triangle is $A = \frac{1}{2}bh$.

2. Pose problem involving area of a triangle other than a right triangle

Note: Before proceeding, make sure pupil understands what is meant by the height or altitude of a triangle (Grade 8 Mathematics, p. 56).

Mr. Jack bought a piece of ground at the intersection of two streets. It was triangular in shape. The frontage was 75 ft. It was 50 ft. deep. (The depth is the perpendicular distance to the front.) How many square feet of property did he buy?





a. Have pupil question whether formula $A = \frac{1}{2}bh$ can be used in the solution, since this is not a right triangle.

b. Have him proceed as in 1-b and 1-c to build up a rectangle about the △.

c. Have pupil develop the relationship between areas of the △s and the area of the rectangle.

1) Have teacher demonstrate (using pre-cut materials) by superimposing △ II on △ I and △ IV on △ III, that the area of the original is one-half the area of the rectangle. ($\frac{1}{2}$ of 3750 = 1875 sq. ft.)

2) Recommendation: Elicit from pupils

△ I = △ II

△ III = △ IV

Therefore: △ I + △ III = △ II + △ IV

△ I + △ III = $\frac{1}{2}$ area of the rectangle

or △ I + △ III = $\frac{1}{2}$ bh (area of rectangle = bh)

d. Have pupil compute using formula.

| Area of Rectangle | Area of Triangle |
|---|---|
| $A = bh$ | $A = \frac{1}{2}bh$ |
| $A = 50 \times 75$ | $A = \frac{1}{2} \times 50 \times 75$ |
| $A = 3750$ sq. ft. | $A = 1875$ sq. ft. |

e. Conclude: the formula for computing the area of any triangle is: $A = \frac{1}{2}bh$.

3. Have pupil find area of several triangles using the formula.

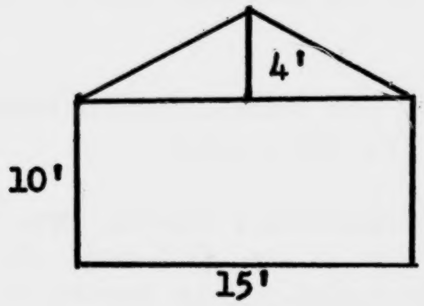
4. Have pupil find area of irregular figures (triangles and rectangles combined).

Illustrative problem:

Find the surface area of the front of the garage as illustrated below.

Suggested solution:

a. Have pupil see that the front view of the garage is composed of two figures: a rectangle and a triangle. Its total area can be found by adding the area of the rectangle to the area of the triangle.



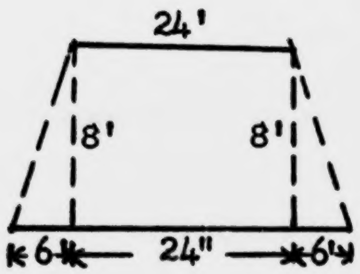
| **Area of Rectangle** | **Area of Triangle** |
|---|---|
| A = bh | $A = \frac{1}{2}bh$ |
| A = 15x10 | $A = \frac{1}{2} \times 15 \times 4$ |
| A = 150 | A = 30 |

Total area 150 + 30 = 180

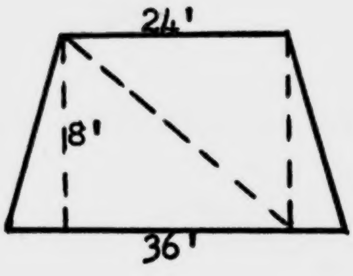
Therefore, the area of the front view of the garage is 180 sq. ft.

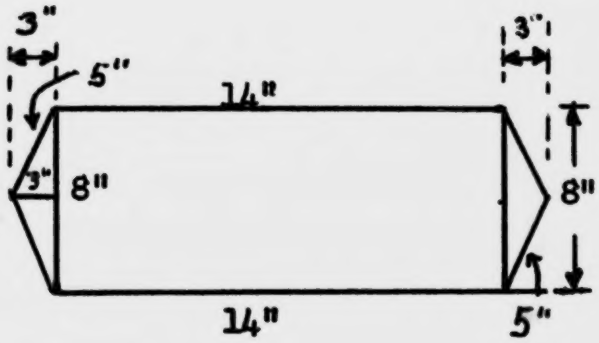
B. Suggested practice

1. Find the areas of the following figures:



or





C. Specific outcomes

1. The area of a △ is one-half the area of a rectangle with an equal base and equal altitude.

2. The area of a △ may be computed by using the formula: $A = \frac{1}{2}bh$.

3. The total area of some irregular figures can be found by dividing the figure into rectangles and triangles, computing the area of each figure, and finding the sum of the areas.

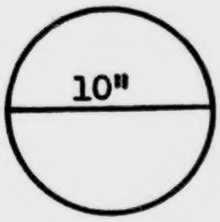
## VII. Using the Area of a Circle

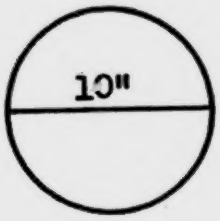
### A. Suggested procedure

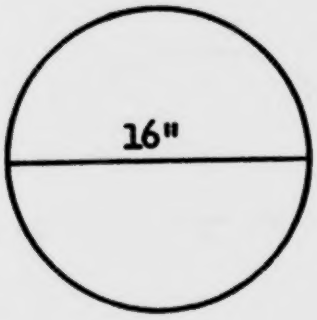
Illustrative solutions:

Pose problem

A group of junior high school students went to the Appian Way Pizzeria after a ball game. The boys asked, "Would we have more to eat if we bought two $1 pies having 10 inch diameters, or one $2 pie having a 16 inch diameter?"







1. Have pupil realize that he needs to know how to find the area of a circle.
2. The area of a circle: $A = \pi r^2$.
3. Review meaning of $\pi$

The circumference of a circle is a little more than 3 times the length of a diameter. $\pi$ is approximately $3 \frac{1}{7}$ $(\frac{22}{7})$ or, expressed decimally, 3.14. The circumference is approximately 3.14 times the length of a diameter.

4. Compute the areas of the pizzas

| Area: Circle 10" in diameter | Area: Circle 16" in diameter |
|---|---|
| $d = 10$; $r = 5$ | $d = 16$; $r = 8$ |
| $A = \pi r^2$ | $A = \pi r^2$ |
| $A = 3.14 \times 5 \times 5$ | $A = 3.14 \times 8 \times 8$ |
| $A = 78.50$ sq. in. | $A = 200.96$ sq. in. |
|  | $A =$ about 201 sq. in. |

2 pizzas = 2 x 78.5 = 157.0 sq. in.

Area of 2 - 10" pizzas is 157 sq. in.; area of the larger pizza is about 201 sq. in.

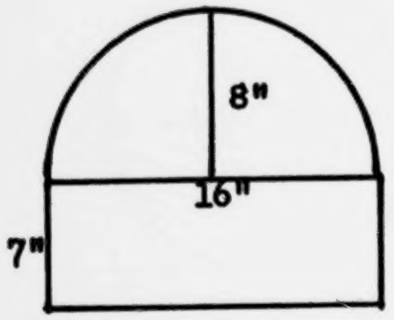
Which would have been the better choice? How much more pie would you get?

5. See texts for additional problems.

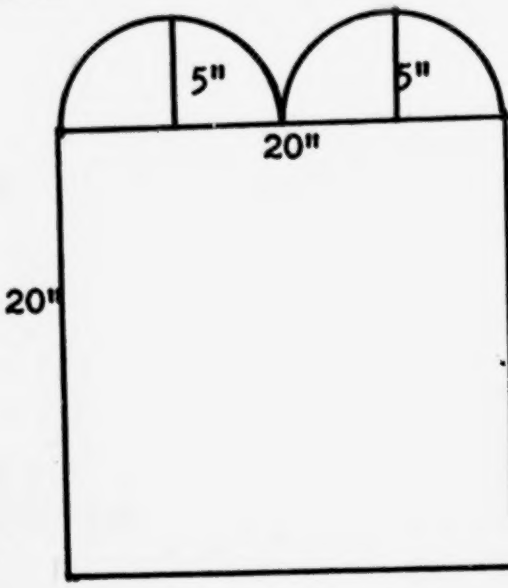
B. Suggested practice

1. Find the area of the following figures:

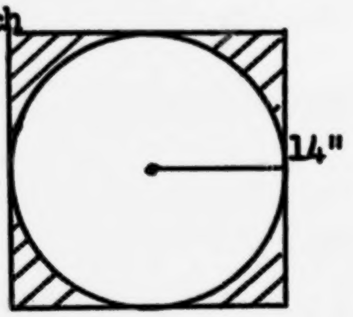
a. Window



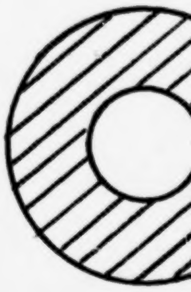
b. Decoration



2. A stamping machine stamps out circles from a square of metal. (See diagram) The area that is shaded is wasted. How much of the metal is waste?



3. A special washer is made from a circle 4 inches in diameter. A small circle with radius one inch has been cut out of the larger one. What is the area of the shaded portion?



C. Specific outcomes

1. The formula used in computing the area of a circle is $A = \pi r^2$.
2. The common approximate values used for $\pi$ are $\frac{22}{7}$ and 3.14.

## VIII. Development of the Concept of Volume

A. Suggested procedure

1. The meaning of size of a 3 dimensional figure

Pose problem

I saw a sign in a restaurant, "Occupancy by more than 125 people is unlawful."

On what basis was this number decided upon? (size of room)

a. What determines the size of a room? (length, width, height)

b. Have pupils learn that the Fire Department has ruled that there must be 400 cubic feet of space for every adult, and there must be 200 cubic feet of space for each child up to 12 years of age.

c. What is the unit of measure the Fire Department is using in alloting space for each person? Answer: cubic foot

d. What is this unit measuring? (space in the restaurant)

e. Why not use a foot or a square foot to measure space? (3 dimensions)

f. What kind of unit is needed to measure space?

2. Units to measure size of 3 dimensional figure

a. Have pupil consider how to describe the size of a box accurately.

1) Box containing 1 pound of a heavy metal; another containing 1 pound of sugar

a) weight - does this tell size?
compare size of both boxes
b) other instances

Conclusion: weight of contents does not tell size.

2) Have pupil fill a carton with books of various sizes or child's building blocks.

Why does not number of books or of blocks tell size? (no uniformity in size)

3) Have pupil recall a grocery carton packed with one kind of boxes of crackers.

a) Why does not the number of boxes of crackers tell size of box? (size of cracker box not known) Smaller cracker box - more boxes; larger cracker box - fewer boxes.

b) How is this unit of measure (the cracker box) better than the books? (uniform size)

c) What is the shape of this unit? (rectangular solid) Express the size of the box in terms of the rectangular solid. (cracker box)

d) Do you think a rectangular solid would serve well as a unit of measure? Good, but different in dimensions. (rectangular solids differ in dimensions)

e) Have pupils recall same size cartons filled with other size cracker boxes and discuss in light of d) above.

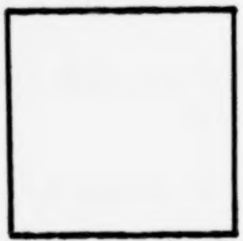
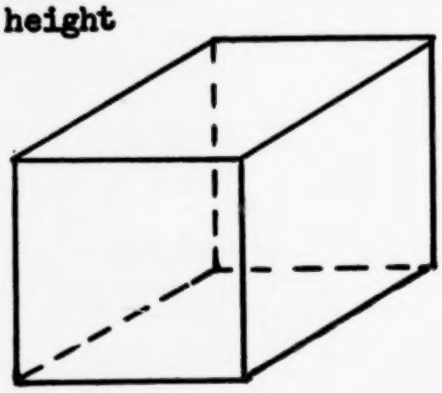
4) Show pupil box of sugar "dots." Tell size by number of "dots." (Box contains 198 dots) Shape of unit (cube); advantage of this shape - all dimensions equal.

b. Have pupil conclude that the unit of measure of space must be a uniform unit and that a cube would be a good unit because all dimensions are equal.

c. Discuss with pupils standard units of measure of space.

1) Elicit names of units of measure of space of which they have heard, cubic inch, cubic foot, cubic yard.

2) Develop relation of these units to linear and square units.

| Linear | Square | Cubic |
|---|---|---|
| 1 dimension:<br>length | 2 dimensions:<br>length<br>width | 3 dimensions:<br>length<br>width<br>height |
| |  |  |
| 1 inch<br>1 foot<br>1 yard | 1 sq. inch<br>1 sq. foot<br>1 sq. yard | 1 cu. inch<br>1 cu. foot<br>1 cu. yard |

3) Show same relation for foot, square foot, cubic foot; yard, square yard, cubic yard.

4) Have pupil give an instance where each cubic unit might be used:

cubic inch - small box
cubic foot - volume of a refrigerator; freezer
cubic yard - earth removed to make room for foundation of building; dump trucks

B. Specific outcomes

1. A unit of space is needed to measure space.
2. A unit of space must be 3 dimensional (length, width, height).
3. A cube is a good unit because the three dimensions are equal.
4. Some units of space measure are: cubic inch, cubic foot, cubic yard.
5. Square units and cubic units are built upon the corresponding linear unit.

C. Suggested practice

1. Keeping in mind the rules of the Fire Department, can you explain why a school auditorium is generally 2 floors high?

2. What other public buildings where many people sit or stand close together can you name? What have you noticed about the height of the ceilings of these rooms? Why is this so?

3. Can you name a household appliance which states in cubic feet how much it can hold?

4. List an object whose size would be given in cubic inches, in cubic feet, in cubic yards.

IX. Experimentation

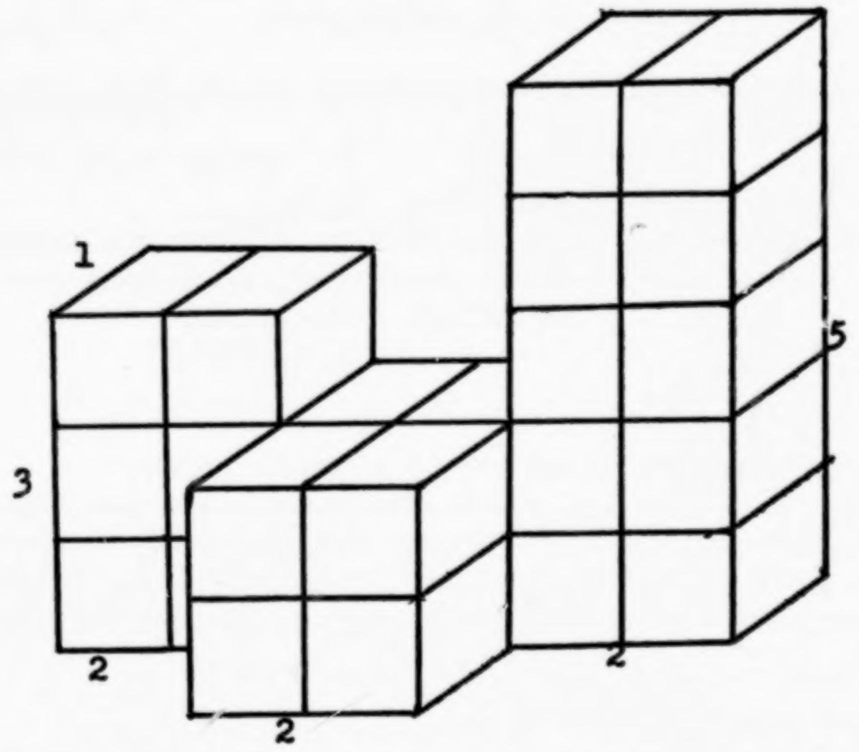
A. Suggested procedure

1. Have pupil discover the meaning of volume by building figures.

   a. Provide pupils with cubic inches (paper or wood pupil-made cubes; shop teacher's demonstration model cubes - 1" x 1" x1"). (See: G-1 Supply List for this item)

   b. Have each pupil take a number of such cubes (e.g. 24) and arrange them to make a figure. (See 1-c)

   c. Have pupil note that each pupil's figure, though different from the others, occupies 24 cubic inches of space.



d. Have pupil see that each section of his figure is a rectangular solid.

e. Have him learn that the number of cubic units in the figure is called the volume of the figure.

2. Have pupil discover the meaning of volume by measuring figures.

   a. Elicit from pupil suggestions for measuring the capacity of a small box (integral dimension).

   b. Have him find the number of cubic inches it contains by filling it with inch cubes.

   c. Have him work similarly with the small boxes.

   d. Have him see that he has measured the capacity of each box in terms of a cubic unit.

   e. Have him conclude that he has found the volume of each box.

B. Specific outcomes

1. The space a rectangular solid occupies is called its volume.
2. The capacity (how much it holds) may be considered its volume. (inside measurement)
3. The volume or capacity (space) is measured in cubic units.

## X. Development of the Formula for the Volume of Rectangular Solid

### A. Suggested procedure

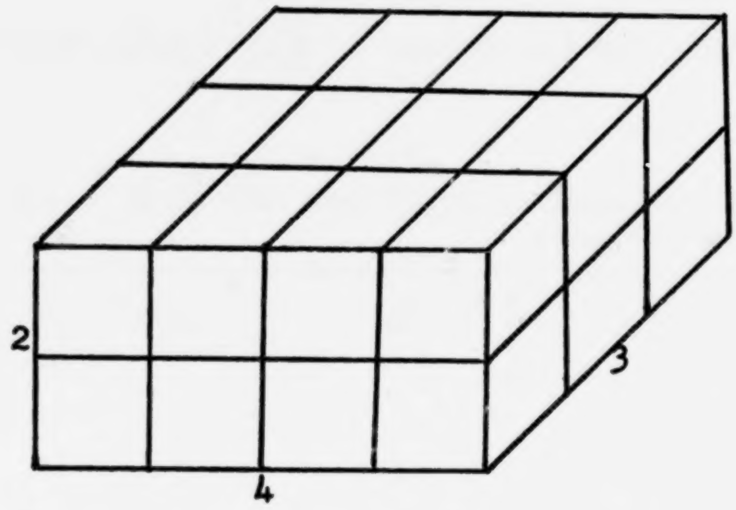
1. Have pupil make a rectangular solid one cube high using the 24 cubic inches as follows:

| Height | Width | Length | Width | Length | Volume |
|---|---|---|---|---|---|
| 1 cube | 1 cube | 24 cubes (show length is 24 inches) | | | 24 cu. in. |
| 1 cube | 2 cubes | 12 cubes (12 inches) | 2 rows, 12 cubes in a row | | 24 cu. in. |
| 1 cube | 3 cubes | 6 cubes (8 inches) | 3 rows, 8 cubes in a row | | 24 cu. in. |

Have pupil conclude that a rectangular solid 1 cube (1 cubic inch) high has a volume equal to the number of inches in a row (length) multiplied by the number of rows (width).

2. Have pupil make rectangular solids 2 inches high using the 24 cubic inches.

| Height | Width | Length | Bottom Layer (length) (width) | Number of Layers (height) | Volume |
|---|---|---|---|---|---|
| 2 cubes (inches) | 1 cube (inch) | 12 cubes (12 inches) | 12 cubes | 2 | 24 cu. in. |
| 2 cubes (2 inches) | 2 cubes (2 inches) | 6 cubes ( 6 inches) | 12 cubes | 2 | 24 cu. in |
| 2 cubes (2 inches) | 3 cubes (3 inches) | 4 cubes ( 4 inches) | 12 cubes | 2 | 24 cu. in. |



a. Similarly, with width 4, 6, 12.

b. Have pupil conclude that the volume of a rectangular solid 2 layers high is equal to 2 times the number of cubes in the bottom layer.

c. Have pupil state this as a formula:

1) 2 times the number of cubes in the bottom layer
2) height (h) times length times width (l x w)
3) V = h x l x w
4) V = lwh

3. Have pupil verify the formula by experimenting with other combinations of dimensions.

4. Solids whose dimensions contain fractions.

a. Have pupil measure a line and note that the measure can contain a fractional part of a unit, e.g., 6½ in.

b. Have pupil draw a rectangle (e.g., 3½" x 2½") on graph paper.

1) Have him note that the area contains a fractional part of a square inch. ($8\frac{3}{4}$ sq. in.)

2) Have him see that he can find the area more easily by using the formula A = bh.

c. Have pupil consider the problem of a box whose dimensions are not an integral number of units. (He may have had that experience in experiment II-A-2)

1) Have pupil fill such a box with inch cubes.
2) Have him note the space remaining (not enough for whole cubes but enough for parts of cubes).
3) Have him suggest ways of measuring that unfilled space. (parts of cubes)
4) Have him conclude that the easiest way to find the volume is by using the formula V = lwh.

B. Specific outcomes

1. To find the volume of a rectangular solid, multiply the number of units in the length by the number of units in the width, by the number of units in the height.

2. The formula for the volume of a rectangular solid is $V = lwh$.

C. Practice

1. A schoolroom is 35' long, 20' wide and 14' high.

   a. How many cubic feet of air space does the room contain? (diagram on graph paper)

   b. If the weight of air is $1\frac{1}{4}$ oz. per cubic foot, what is the weight in ounces of the air in the room?

   c. What is the weight in pounds?

2. Refer to question 1: Allowing 200 cubic feet of air space per child, what is the maximum number of children that may be assigned to a room?

3. How many cubic feet are there in a cubic yard? Experiment with inch cubes each representing a cubic foot and fill a 3 inch cube.

4. A cubic foot of water weighs 62.5 lbs. What is the weight of water in a fish tank 3' long, 1' wide, and 1' high?

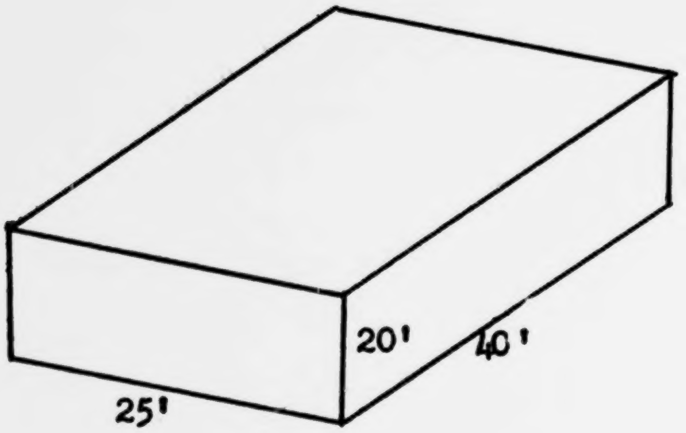
5. Bob's father bought crushed rock for his driveway which is 90 ft. by 12 ft. He will make his driveway 4 inches ($\frac{1}{3}$ ft.). How many cubic feet of rock will he need? Such rock is generally bought by the cubic yard. (27 cubic feet equals 1 cubic yard) How many cubic yards will he buy?

6. A power shovel can pick up one cubic yard of earth at a time. How many scoops will be needed to fill an earth removal truck which measures 18' x 6' x 3' ?

7. In many Westerns, we see a stagecoach thief lightly throwing a chest of gold off the top of the stagecoach. If such a chest measures $2\frac{1}{2}$ feet by 2 feet by $1\frac{1}{2}$ feet, and one cubic foot of gold weighs about 1200 lbs., how many pounds of gold is that thief tossing off so lightly? How many tons is that?

8. How many cubic inches are there in 1 cubic foot?

9. What will it cost to build a house this size at $2 per cubic foot?



10. How many cubic feet in one cubic yard?

11. In the ninth year science room there is a rectangular fish tank. Have pupil measure its dimensions. Have him find the volume and the capacity in gallons. (231 cu. in. = 1 gal.)

12. What is the weight of the water filling the fish tank, (#11), if a gallon of water weighs approximately 8.3 lbs.?

## XI. Volume of a Cylinder

A. Suggested procedure: If necessary, the teacher may review the treatment given this subject in Mathematics Grade 8, pages 116-118.

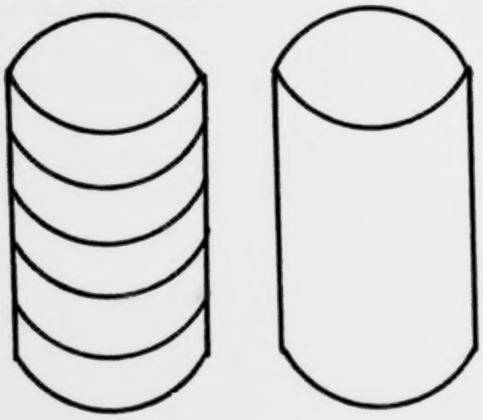
1. Motivation

Have pupil consider how to find the volume of a can (cylinder) e.g., 2 inches in diameter and 5 inches high.

a. Have pupil recall that the cubic inch is a unit of measure of volume.
b. Have pupil try to fill the can with cubic inches and have him see that much space is left unfilled. Does this mean that the volume cannot be measured in cubic inches?
c. Have pupil fill the can with sand. Have him pour this sand into a box and see that the volume of the can can be expressed in cubic inches. (The sand will about fill a box 2x2x4", i.e., will measure about 16 cubic inches.)
d. Have him see that the volume of the can (cylinder) is about 16 cu. in.
e. Have pupil see that this is not a practical way to find the volume of a cylinder and that a shorter way is desirable.
f. Elicit the need for a rule or formula.

2. Developing the formula for the volume of a cylinder

a. Have pupil see that as in the rectangular solid, the volume can be found by multiplying the number of cubic inches in the bottom layer, B, by the number of layers, h. In the drawing, h is 5.



b. Have pupil recall that in the rectangular solid the number of cubes in the bottom layer (B) was equal to the number of units in the length, multiplied by the number of units in the width, i.e., the area of the base.

c. Have pupil recall that the area of the base of a cylinder- a circle- is $\pi r^2$.

d. Therefore, the volume of the cylinder equals the number (B) of cubic inches in the bottom layer, multiplied by the number of layers (h), or

$V = Bh$ or since $B = \pi r^2$

$V = \pi r^2 h$

e. Test the formula to find the area of the can in 2.

$V = \pi r^2 h$

$V = \frac{22}{7} \times 1 \times 1 \times 5 = \frac{110}{7}$

$V = 15\frac{5}{7}$

The volume of the can is about 16 cubic inches. (This is what the pupil found the volume to be when measuring the volume with sand.)

B. Practice

1. The base of a water tank is a circle with a diameter of 14 feet. What is the area of the base?

2. The tank in example 1 is 8 feet high.

   a. What volume of water does it contain?
   b. One cubic foot of water holds 7.5 gallons. How many gallons of water does the tank hold?

3. A water pipe has an inside diameter of $3\frac{1}{2}$ ft.

   a. How many cubic feet of water does the pipe, 32 feet long, hold when full?
   b. How many gallons is that? (1 cu. ft. = 7.5 gal.)

4. In many countries, people carry water in cylindrical buckets from the village pump to their homes. Such a bucket measured 10 in. in diameter, and will filled to a height of 14 in.

   a. How many cubic inches of water does this bucket hold?
   b. Does the bucket hold more than one half of a cubic foot of

water? (1728 cu. in. = 1 cu. ft.)

c. If a cu. ft. of water weighs 32.5 lbs., the water in the bucket weighs more than _____ lbs. (more than 32 lbs.)

5. Have pupil notice the water tanks on buildings. Have him note that the circular steel bands are placed closer and closer as they near the bottom. Why? (pressure of water) A water tank is 14 ft. in diameter and 20 ft. high.

    a. What is the volume of water in the tank?
    b. What is the weight of the water in the tank? (1 cu. ft. of water weighs 62.5 lbs.)

6. Additional practice: Compute volumes of actual cans as used in the home.

C. Specific outcomes

1. The volume of a cylinder cannot be found easily by measuring directly.

2. A practical way to find the volume of a cylinder is to compute it.

3. The formula $V = \pi r^2h$ gives the rule for computing the volume of a cylinder.

## XII. Finding the Surface Areas of Solid Figures

Note: In this section, the previously developed concepts and formulas will be combined and employed in computing surface areas of various solids. Remind pupil of the multitude of shapes and sizes encountered in daily life.

A. Suggested procedure

1. Identify solid figures (objects or pictures). Use models, book, orange, etc.

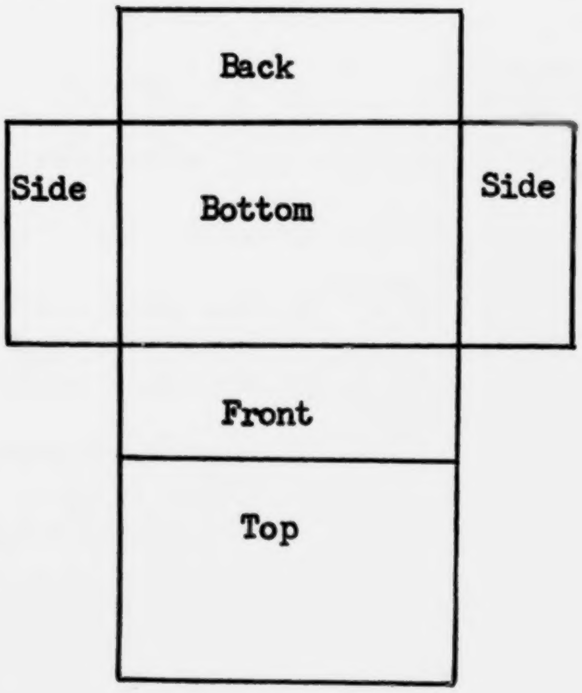
    a. rectangular solid - 6 rectangular faces

    b. cube - 6 square faces

    c. square pyramid - 1 square face, 4 triangular faces

    d. cylinder - 2 circular bases and a curved surface

2. Show pupil cardboard packing box and lead pupil to suggest how it



can be cut open into a flat surface.

3. Have him note the pattern formed: 3 pairs of equal rectangles (front and back, top and bottom, side and side).

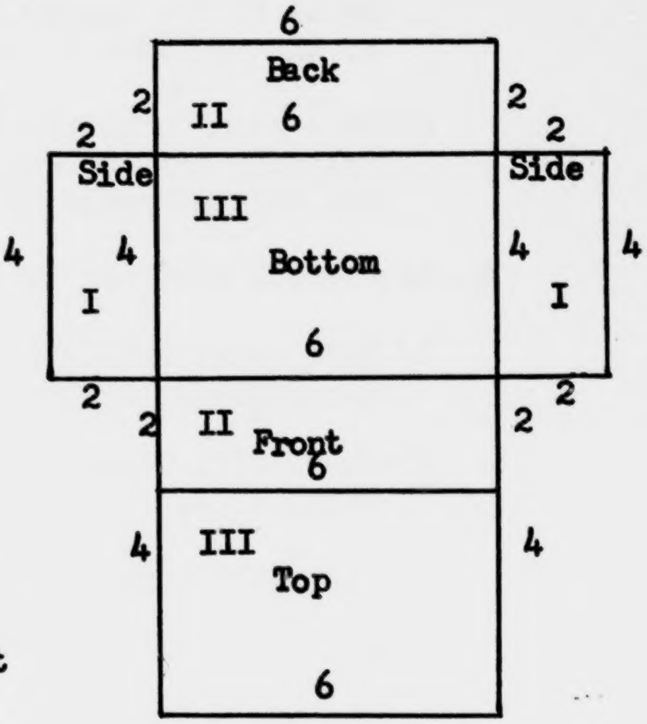
4. Have pupil find the total surface area of a rectangular solid.

   Pose problem

   The Colorful Crayon Company uses boxes 6" x 4" x 2" to package their crayons. How many square inches of cardboard are used to make each box?

5. Have pupil draw the pattern and put in the dimensions.

   By inspection, have pupils name the rectangles with equal areas.



| Area of Front and Back | Area of Top and Bottom | Area of Sides | Total Area |
|---|---|---|---|
| A = bh | A = bh | A = bh | 24 + 48 + 16 = 88 sq. in. |
| A = 2 x 6 | A = 4 x 6 | A = 2 x 4 | |
| A = 12 | A = 24 | A = 8 | |
| 2 x 12 = 24 | 2 x 24 = 48 | 2 x 8 = 16 | |

88 sq. in. of cardboard are used to make each box.

Note: Have pupil realize that there are alternate methods that may be used to find the total area.

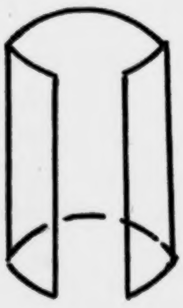
6. In a similar manner, find the total surface area of a cube.

7. Have pupil find lateral surface area of the cylinder (not including bases).

   a. Experiment with a piece of paper, or remove a label from a can.

      1) Have pupil roll a piece of paper into a cylinder. (6"x9")

      2) Have pupil note that the lateral surface (the curved surface) of the cylinder is equal to the surface of the paper. (The paper is a rectangle in form.)



3) Have pupil observe that the circumference of the cylinder is equal to the length of the rectangle (9 inches). The height of the cylinder is equal to the width of the rectangle.

4) Have pupil conclude that the area of the lateral surface of a cylinder can be found by multiplying the number of units in the circumference (9) by the number of units in the height (6).

b. Have pupil find the lateral surface by computing the circumference.

1) Have pupil recall that the circumference of a circle is found by multiplying the diameter by $\pi$. ($c = \pi d$)

2) Have pupil measure the diameter of a can, e.g., 2 inches in diameter and 7 inches in height.

3) Have pupil compute the circumference.

$d = 2$

$c = \pi \times 2$

$c = \frac{22}{7} \times 2$

$c = 6\frac{2}{7}$

The circumference is $6\frac{2}{7}$ inches.

4) Have pupil compute the area.

$A = bh$ (The lateral surface can be considered a rectangle.)

The base is the circumference of the circle = $6\frac{2}{7}$ inches.

$A = \frac{44}{7} \times 7$

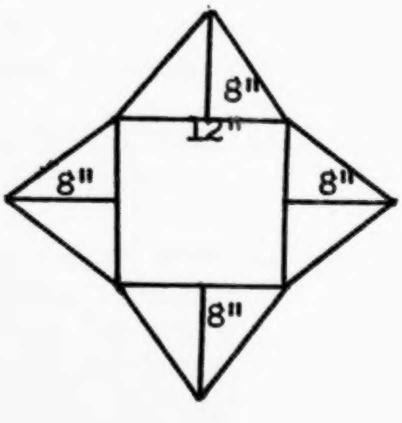
$A = 44$

The lateral surface of the can is 44 sq. in.

Optional:

8. Have pupil find the total area of a square pyramid.

a. Have pupil see that the total surface of a square pyramid consists of the surface of the base which is a square, and the surface of 4 equal triangles (isosceles).



b. Have pupil compute the number of square inches of surface of the square pyramid.

| **Area of Square** | **Area of Triangles** |
|---|---|
| $A = bh$ | $A = \frac{1}{2} bh$ |
| $A = 12 \times 12$ | $A = \frac{1}{2} \times 8 \times 12$ |
| $A = 144$ | $A = 48$ |
| | 4 triangles = $4 \times 48 = 192$ |

Total surface of square pyramid: $144 + 192 = 336$ sq. in.

B. Specific outcomes

1. The total surface of certain solids can be shown as patterns on a plane.

2. The total surface area of solids which have plane faces is equal to the sum of the areas of the faces.

C. Suggested practice

1. A metal smokestack 7 feet in diameter and 30 feet high is to be painted. How many square feet of surface will be painted? If one gallon of paint is needed for 300 square feet of surface, how many gallons will be needed?

2. A canvas tent has the shape of a square pyramid. The side of the square is 10 feet and the altitude of the triangular face is 10 feet. How many square feet of canvas are there in the tent?

3. How much metal is needed for the surface of a cylindrical can (exclude bases), which is 14 inches in diameter and 21 inches high? How much metal is needed for the two circular bases? Total area?

# IV.

# CONSUMER MATHEMATICS

## I. Computing Wages

Suggested procedure

A. How employees are paid

1. Have pupils tell of ways they earn money, i.e., baby-sitting, delivering papers, errands, etc.

2. Have pupils explain how their wages are computed, i.e., by the hour, job, etc.

3. Using a chart similar to the following, have pupils suggest types of workers in each payment category.

| Method of payment of employees | Type of Work |
|---|---|
| Salary | Teacher, etc. |
| Hourly wage | Baby-sitter, etc. |
| Piece work | Tailor, etc. |
| Commission | Salesman, etc. |
| Fees | TV repairman, etc. |

4. From the chart, elicit from pupils that workers are paid in various ways. Also that the type of work usually determines the method of payment.

B. How wages are computed

1. Salary

a. Lead pupils to understand that a salary is a fixed sum of money paid for a definite period - usually a week, a month, or a year. As a rule, the worker on a salary does not receive money for overtime.

Note: In all of the suggested practice, the examples are given in a graded form. The initial problem may seem very easy to do. They are presented for the purpose of helping pupils to determine the method of solution. The teacher should provide additional examples of each type for reinforcement. Such additional examples are found in the textbooks.

b. Suggested practice

1) Mr. Moses receives a salary of $450 per month. How much does he earn per year?

2) Harry's father earns $78 per week. John's father earns $275 per month. Whose father earns more at the end of the year? How much more?

3) Mr. Jones received a salary of $5,600 per year. What is his salary per month? What is his salary per week?

Note: Have pupil realize that you cannot find the weekly salary by dividing the monthly salary by 4.

4) John works in a grocery store for the summer and earned $37 per week. How much did he earn if he worked for 9 weeks?

2. Hourly wages

a. Mary receives 75¢ per hour as a baby-sitter.

How much does she earn in 4 hours?
How much does she earn in 5 hours?

If any part of an hour must be paid for as a full hour, how much will she receive for 3½ hours; for 4¼ hours?

b. In certain occupations, the indicated wage rates apply. The employee is paid only for actual time worked. Complete the chart.

| Hours | Rates | Wages |
|---|---|---|
| 6 | 85¢ | |
| 7½ | 90¢ | |
| 4 | $1.25 | |
| 5 hrs., 20 min. | $1.50 | |

c. Elicit from pupils that such wages depend upon the number of hours worked and the rate per hour.

Would you prefer to be paid salary or wages? Why?

1) Angela worked as a baby-sitter for Mrs. Diaz from 8 p.m. to 11:20 p.m. at 80¢ per hour. How much did Angela earn?

2) John works as a delivery boy after school. He is paid at the rate of 75¢ per hour. If he worked 20 hours last week, what were his total earnings? How much would he earn in 18 hours at 85¢ per hour? How much would he earn in 12 hours at 95¢ per hour?

3) Bill works as a soda fountain helper in a drug store. He works 3 hours each day. If he works 5 days during the week, how much will he earn for the week if he receives 80¢ per hour?

d. Tell pupils that in most trades employees are paid "time and one half" ($1\frac{1}{2}$) for overtime. Overtime in this sense is any time in excess of the number of hours per day stated in the union contract.

1) Have pupils complete the following chart:

| Hours | Rate | Regular Wages | Overtime in hours | Overtime Wages | Total Wage |
|---|---|---|---|---|---|
| 10 | $1.00 | $8.00 | 2 | $3.00 | $11.00 |
| 10 | $1.20 |  | 2 |  |  |
| 9 | $1.40 |  | 1 |  |  |

2) Mr. Wright is an electrician. He receives $4.20 per hour. The union contract calls for a six-hour day. Electricians are paid time and a half for overtime. How much will he make if he works 8 hours per day?

3) Mr. Spring is an electrician. His pay is $4.20 per hour for 6 hours and time and a half for overtime. Find the weekly earnings if he worked the following number of hours one week:

Monday - 6 hours
Tuesday - 7 hours
Wednesday- 8 hours
Thursday - 7 hours
Friday - 6 hours

4) (a) Mr. Smith earned $68 for 4 days work. Find his daily rate of pay.

   (b) Mr. Cuomo earned $91 for 3½ days of work. What was his rate per day?

3. Piece Work

a. Development

1) Joseph works in a key shop, where he receives 8¢ for each key he makes. If he makes 160 keys a week, how much money will he receive for his work? If he makes 280 keys? If he makes 340?

2) Mary earns 12¢ for each sleeve she sews. How much does she earn by sewing 96 sleeves per day?

   Lead pupils to see that such wages are based on the number of pieces completed and not on the time spent doing such work.

3) Louis and Tom work in a shoe factory, attaching heels to the shoes. They receive 15¢ per pair. Last week, Louis finished 750 pairs of shoes, and Tom finished 675 pairs. How much more than Tom did Louis earn?

4) To earn at least $20 per day, how many pairs of shoes must Tom heel?

   Discuss briefly the role of the union in fixing "piece work" rates.

b. Suggested practice

1) James sold newspapers to 32 homes. He was paid 40¢ a month per home. How much did James earn per month?

2) Harry's father owns a car wash business. He charges $1.25 per car. He and his helpers washed 280 cars. What was the total charge for washing the cars?

3) Anna works in a blouse factory. She is paid 6¢ per blouse for each one she irons. If she can iron an average of 260 blouses per day, how much will Anna earn per day? How much will she earn for a five-day week?

4. Commission

a. Development

1) Tommy sells boxes of cookies and earns 15% of all his sales. During the past week he sold $85 worth of cookies. How much did he earn?

2) Mary sells cosmetics and is paid a commission of 40%. Last week she sold 23 compacts at $1.98 each, and 19 lipsticks at $1.29. How much did she earn?

3) Mr. Mandel, a real estate salesman, receives $50 per week, plus a 5% commission on all his sales. Last week he sold a house for $15,000. How much did he earn last week? This week Mr. Mandel made no sales. How much did he earn?

Elicit from pupils that workers whose wages are based solely on commission have no fixed income. Discuss briefly the advantages and disadvantages of the salary plus commission form of payment over straight commissions.

4) Mr. Cohen works in a shoe store. He earns $85 per week and 6% of all sales over $750. Last week he sold $900 worth of shoes. How much did he earn?

b. Suggested practice

1) Mr. Schwartz drives a taxi and receives 45% of his collections as a commission. Last week his collections totaled $190. How much did he earn?

2) Miss Marker was employed by a company to sell cosmetics from door to door. She was paid 40% of her sales as a commission. Find her commission for a week when her sales were $155.

3) Mr. Morris is an insurance agent. He is paid a salary of $60 per week plus a 3% commission on all insurance premiums he collects for policies which are already in force. Last week Mr. Morris collected $985 in premiums. What was his commission?

4) Mr. Morris also is paid 30% of the premium on all new policies. Las week he collected $325 in premiums on new policies. What was his commission?

5) What were his total earnings?

C. How self-employed people earn their money

1. Fees: Guide the pupil to an understanding of the meaning of a fee. When payment is made to a person for his special skill or knowledge, it is called a fee. Usually, people who are paid a fee give you their services for a limited time. Most professional people such as doctors, lawyers, dentists, etc., are in this category. A similar situation exists among repair men.

   Present the problem:

   a. When Harry had a deep cut on his arm, his mother took him to the office of Doctor Morris, the family doctor. She paid the doctor $5 for this visit. Later, Harry required an operation on his leg. Dr. White, the surgeon, charged Harry's mother $500.

      Explain why Dr. White received more than Dr. Morris.

   b. When Mr. Oxen sold his house, he paid a lawyer $100 to check the transaction. However, when Mr. Oxen later was sued for a breach of contract, he paid another lawyer $1,000 to defend him.

      Explain why Mr. Oxen was willing to pay the second lawyer so much more for his services.

2. Profit

   Note: Pupils have studied the topic, "Profit and Loss" in the previous grades. At this point, we wish the pupils to understand that a business man's earnings which come from profits are similar to the wages of employees.

   Mr. Rodriguez owns a vegetable store. His gross profit for the month of May was $582.00. The "overhead" for carrying on his business (rent, light, heat, telephone, etc.) during the month was $170. What was his net profit?

II. Deductions from Wages

A. Social Security

1. Present the problem: During the summer, Jose earns $1.25 an hour working as a helper in a store. He works 40 hours per week.

   a. How much does he earn per week?
   b. How much of this do you think he "takes home"?

2. Through discussion elicit from pupils that "gross earnings" and "take home" pay are not necessarily the same. Lead pupils to realize that required and voluntary deductions reduce the gross earnings.

3. Have pupils learn the meaning of Social Security as a family insurance plan sponsored by our Federal Government.

   Note: Use available Social Security information and/or textbooks.

4. Present the following questions: (referring to example 1, above)

   a. At the rate of 3%, how much will be deducted for Social Security from Jose's weekly salary?

      Note: The 3% rate (1960-61) is used to avoid fractional per cents.

   b. Every employee's Social Security payment is matched by an equal amount from the employer. What is the total weekly tax paid to the government for Jose's Social Security?

   c. Jose's father earns $400 a month. What is the monthly deduction for Social Security at 3%? How much will Jose's father earn in one year? How much will be deducted for the Social Security tax for one year? How much will his employer pay towards Jose's father's Social Security?

   d. Only the first $4,800 earned in a year is taxed for Social Security. At 3%, how much tax does Jose's father pay in 1 year if he earns $5,400? How much does his employer pay? What is the total yearly payment credited by the government to Jose's father for Social Security?

   *e. Jose's uncle owns a grocery store. Because he is self-employed, his tax rate is 4½% on the first $4,800 earned. What tax would Jose's uncle pay if the store earned $4,200? $4,400? If the store earned $6,400?

5. Suggested practice

   Below is a chart showing the Social Security tax rates that have been scheduled by Congress. Use this chart to answer the following exercises. (Remember that these rates are applied to the first $4,800 of annual income.)

| Calendar Year | Wage Earner Pays | Employer Pays | Self-employed Person Pays |
|---|---|---|---|
| 1960-61 | 3% | 3% | $4\frac{1}{2}\%$ |
| 1962 | $3\frac{1}{8}\%$ | $3\frac{1}{8}\%$ | $4\frac{7}{10}\%$ |
| 1963-65 | $3\frac{5}{8}\%$ | $3\frac{5}{8}\%$ | $5\frac{2}{5}\%$ |

a. In 1961, Maria earned a weekly salary of $55. How much was deducted from Maria's salary for social security each week? How much was contributed by Maria's employer each week towards Maria's accumulations for social security?

b. Fred's salary was $265 for each month in 1960. How much was deducted for social security each month? What was the contribution of Fred's employer to social security on behalf of Fred each month. What was the total amount contributed by Fred and his employer for the month of February, 1960?

c. In 1962, Thomas earned $4,800. What was the total amount deducted from his yearly salary during 1962? How much was contributed by his employer that year for Thomas? ($3\frac{1}{8}\% = .03125$)

d. In 1962, Thomas' brother earned $5,200. How much did Thomas' brother's deduction for social security amount to? His employer's contribution? Compare your answer with the answer to problem "c". Explain.

e. Mr. Rivera bought a cleaning store in 1961. During the year, he made a profit of $4,400. How much did he pay in social security tax? During 1962, he made a profit of $6,000. How much did he pay in social security tax in 1962? ($4\frac{1}{2}\% = .045$)

B. Withholding Tax

1. Present the situation: The school sponsors such projects as Class Dance, Field Day Prizes, Talent Show, Graduation Prizes.

   a. Name various ways in which the G.O. can raise money.

   b. The G.O. is a student government. How may a civil government raise money?

   c. Elicit from pupils that one means of raising money is by taxing the earnings of individual wage earners.

2. Have pupil understand that if a person works for wages, the employer computes his income tax on an annual basis, withholds the proper amount of money from the employee's wages, and turns it over to the government.

   Note: Pupils should understand that if they work for part of a year (e.g., several weeks during the summer, or before the Christmas holidays), the withholding tax has been figured on an annual basis. Most of them are entitled to a complete refund of such deductions, which must be applied for during the period when income tax returns must be filed. More detailed information is available from the Income Tax Kit or from any I.R.S. office.

3. Help pupil understand that

   a. The system of withholding money from a wage earner's pay is called payroll deduction.

   b. There may be other voluntary payroll deductions, such as union dues, health insurance, and savings.

   c. The employer must furnish the employee with a record of all payroll deductions.

4. Lead pupils to understand the importance of being able to check their wage records, so that they may make sure of the correctness of the W-2 Form which they will receive from the employer at the end of each year.

5. Suggested practice

   a. Using the figures in the table below, compute the salaries in each case and enter the amounts:

| Name | Hours per day | | | | | Total Hours | Rate | Salary (Gross) |
|---|---|---|---|---|---|---|---|---|
| | M | T | W | Th | F | | | |
| Bob | 3 | 5 | 8 | 8 | 8 | ? | $1.57 | ? |
| Mary | 6 | 8 | 7 | 8 | 8 | ? | $1.49 | ? |
| Jose | 7 | 8 | 7 | 8 | 8 | ? | $1.59 | ? |
| Rita | 8 | 8 | 8 | 8 | 8 | ? | $1.68 | ? |
| Tom | 8 | 7 | 7 | 6 | 8 | ? | $1.97 | ? |
| Totals | ? | ? | ? | ? | ? | ? | | |

b. Find the total of hours for each day.

c. Find the total hours for all employees for the week.

d. Check the sums of the daily hours vs. the total hours for the week. What should you find to be true?

6. Take Home Pay

a. Assuming a 40 hour work week, compute the "Gross Salary", "Social Security Tax" at 3%, and then the "Take Home" pay in each of the cases below. Have the pupils enter the results of their computations in the blank spaces.

| Name | Hourly Wage | Total Wage | Social Security Tax | Withholding Tax | Take Home Pay |
|---|---|---|---|---|---|
| J. Butler | $1.60 | | | $9.40 | |
| D. Perez | $1.28 | | | $7.10 | |
| P. Davis | $1.20 | | | $6.80 | |
| J. May | $2.20 | | | $13.70 | |
| T. Green | $1.50 | | | $8.70 | |

b. In the above example the "Take Home" pay was established.

1) Find the total "Take Home" pay for all the employees.

2) The payroll clerk was sent to the bank to get the money to pay the employees.

He returned with the following bills and coins:

| Number | Denomination |
|---|---|
| 2 | Hundred dollar bills |
| 1 | Fifty dollar bill |
| 1 | Five dollar bill |
| 1 | One dollar bill |
| 1 | dime |
| 1 | nickel |
| 1 | penny |

Was this the correct amount needed?

3) Could he pay the employees using the money he brought back from the bank in the denominations above? Explain.

4) How could he find the least number and denominations of bills and coins necessary to correctly fill the pay envelopes of each employee? (currency breakdown)

5) Have pupils fill in the following chart. Check their work.

| Name | Take Home Pay | Denominations | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $10 | $5 | $1 | 50¢ | 25¢ | 10¢ | 5¢ | 1¢ |
| J. Butler | $52.68 | 1 | 0 | 2 | 1 | 0 | 1 | 1 | 3 |
| D. Perez | 42.56 | | | | | | | | |
| P. Davis | 39.76 | | | | | | | | |
| J. May | 71.66 | | | | | | | | |
| T. Green | 49.50 | | | | | | | | |
| Totals | $256.15 | | | | | | | | |

6) This time the payroll clerk used a memo such as the one below before going to the bank.

What would be the advantage of using such a memo?

Fill in the memo.

| Currency Memo | | |
|---|---|---|
| Denomination | Number | Total Value |
| $10 | | |
| $ 5 | | |
| $ 1 | | |
| 50¢ | | |
| 25¢ | | |
| 10¢ | | |
| 5¢ | | |
| 1¢ | | |

7) Find the sum of the "Total Value" column. With what figure should this sum agree?

## III. Taxation

### A. How we support our government

Present the following questions:

1. Through Federal Taxes, the people of America purchase many services such as National Defense and Social Security. Can you think of others?

2. There are approximately 180 million people in the U.S. At $1 per person, how much would this yield?

3. If the budget of the Federal Government were 72 billion dollars, how much should each person pay, if each paid an equal share?

4. Elicit from pupils examples of specific taxes which people are called upon to pay (sales, admissions, luxury, etc.)

### B. Federal income tax

1. Guide the pupil to understand that the individual income tax is based on the principle that each person should contribute his share of the total tax in accordance with his ability to pay.

2. Robert earns $76 per week for 52 weeks of the year. He has a weekly payroll deduction of $11.60 for income tax, $3.10 for social security and group insurance.

   a. What is Robert's weekly "take home" pay?

   b. Find Robert's "gross income" for the year.

   c. In one year, how much income tax will have been withheld?

3. John has the same type of job as his brother, Robert, and also earns $76 a week. However, he is married and has one child. John is said to have 3 "exemptions", his wife, his child, and himself. John's payroll deductions are $6.90 for income tax, and $4.15 for social security and group insurance.

   a. What is John's weekly "take home" pay?

   b. In one year, how much money for income taxes does John have withheld?

   c. Why does John pay less in income taxes than Robert?

4. Once a year a wage earner must fill out an income tax form and send it to the nearest INTERNAL REVENUE BUREAU office. This is called, "filing a return". Obtain some Federal Income Tax forms and instruc-

tion sheets and complete them for a single person whose gross income is $4,800 per year.

5. Tom worked during the summer. He earned $50 per week for 9 weeks. Each week his employer deducted $6.80 as withholding tax. How much did Tom earn for the 9 weeks he worked? Because Tom earned less than the minimum of $600 per year, he is entitled to a refund of all taxes withheld. How much should Tom ask to have refunded when he files his income tax return? (See Note under B, 2 )

C. Sales tax

1. Many cities and states have a sales tax on certain services and retail sales. The sales tax is usually a fixed per cent of the retail price and is added to the retail price.

   a. Find the amount which must be paid for services costing $6.98 if a 3% sales tax is to be added.

   b. Find the sales tax on an article costing $1.50, if the tax rate is 4%.

2. In New York City, the sales tax is 4% (7-1-63). The amount of tax to be paid for retail sales is shown in the table.

| Sale | 1-18¢ | 19-30¢ | 31-55¢ | 56-80¢ | 81-$1.18 | $1.19-$1.30 | $1.31-$1.55 |
|---|---|---|---|---|---|---|---|
| Tax | 0¢ | 1¢ | 2¢ | 3¢ | 4¢ | 5¢ | 6¢ |

Using the table, find the tax to be paid on these sales.

Note: Have pupils note the repeating pattern after $1.00.

| | | | |
|---|---|---|---|
| a. 48¢ | c. $1.23 | *e. $2.25 | *g. $39.98 |
| b. 92¢ | d. $1.76 | *f. $9.98 | *h. $89.90 |

IV. Wise Spending of Wages Earned

A. Opportunities for wise buying

1. Buying at discount

   a. Present problem:

Mrs. Smith bought a steam iron for $11.98 from the C & D department store. Mrs. Brown went down the street to the Royal Discount Store and found that the cost of the same model steam iron was $8.98.

1) What did Mrs. Brown save by buying at the Royal Discount Store? (disregard sales tax)

2) Estimate what per cent Mrs. Brown saved by buying at the Royal Discount Store.

3) What possible reasons could Mrs. Smith have for buying the iron at the C & D Store in spite of the greater cost? (has charge account; didn't do comparison shopping; wished to have the iron delivered; felt the department store would be responsible for possible imperfections)

4) What sales tax did she pay? What was the total cost of the iron to Mrs. Smith?

5) The C & D Department Store charged $1 for delivery of the iron because they said they were selling at a lower price than the list price. Mrs. Smith had the iron delivered to her home in the Bronx. (There is no sales tax on a delivery charge.)

6) How much sales tax did Mrs. Brown pay on the iron? What was the total cost for Mrs. Brown? How much more did Mrs. Smith pay for the iron than Mrs. Brown?

b. Present problem:

The Young Men's Clothing Shop is running a Renovation Sale. The sign on the window reads: "Discount on all suits 30% -50%"

1) What is meant by 30%-50%?

2) Tony chooses a suit with a tag marked: $39.95. The discount on this suit is 30%. If Tony buys the suit, what discount does he get on it? How much will it cost before Sales Tax is added?
Note: Have pupils realize that rounding off to $40 alters the amount by a negligible quantity. Have pupils work with $40.

3) Tony saw another suit marked $69.95. The discount on this suit is 40%. What is the discount? What is the sales price before tax is added?

4) What is the saving on the first suit? On the second suit? What possible reason might Tony have for not taking advantage of the greater saving?

2. Buying at sales

a. Elicit from pupils kinds of sales they have seen advertised, such as: end of season sales, white goods sales, Columbus Day sales, etc.

b. Present problem:

In June, the Family Department Store ran an "End of Season Sale" on Spring Coats. The advertisement announced: "All coats in the Budget Department on sale for $24.95, formerly priced $29.95 to $49.95."

1) What is the saving on a coat originally priced at $37.49? $42.89? $32.95?

2) Mrs. Green bought a coat originally marked $48.89. Her daughter Mary's coat was originally marked $31.49. Another daughter, Joan, bought a coat originally marked $34.95. How much did the family save on the total purchase?

3) Without written computation, what per cent is saved on the $49.95 coat?

4) What is the saving on a coat originally marked $39.95? What per cent is saved?

5) Jessica saw a coat with only the sales price $24.95, on the tag. Why was the original price not entered?

Note: Discuss validity of sales such as "special purchases for this sale." (may be misleading) Using terms:"comparable prices"; "usually sold for"; "regular price" - may be sales gimmicks and untrustworthy.

3. Buying in quantity

a. Present problem:

Two cans of vegetable soup sell for 37¢. At a sale, the L & M Store is selling the same 2 cans for 33¢.

1) Mrs. Lopez has a large family. She buys 6 cans at the sale. How much does she save on 2 cans? on 6 cans?

2) Mrs. Albee,who lives alone, buys one can. How much does she pay for one can at the regular price? (19¢) How much does she pay at the sale price? (17¢) How much does she save by buying at the sale?

3) If Mrs. Lopez bought a can of soup every time she needed one, how much would she pay for the six cans at the regular price? (6 x 19¢)

b. Present problem:

U.S. Maine potatoes sell at 10¢ per pound, 39¢ a 5-lb. bag, 59¢ for a 10-lb. bag.

1) Mrs. Lopez buys a 5-lb. bag of potatoes. Two weeks later she buys another 5-lb. bag. How much more did she pay for

the two 5-lb. bags than she would have paid for one 10-lb. bag. What reason could she have for buying at two different times?

2) Mrs. Rose, a neighbor, has a cool cellar in which she stores perishable items. If she uses potatoes in large quantities, which of the three choices should she make?

3) Mrs. Albee bought a pound every time she needed potatoes. How much will 5 pounds of potatoes cost her? How much more did she pay for 5 pounds than she would have paid if she had bought the 5-lb. bag? Why did Mrs. Albee feel that she really did not lose 11¢ by buying this way?

c. Present problem:

The Smart Shopping Department Store advertised a sale on ladies' hose. The regular price is 99¢ a pair. The sale price is 79¢ a pair, with a box of six pairs at $4.50.

1) What is the saving on one pair of hose?
2) What would six pairs cost at the regular price?
3) What is the saving on a box of 6 pairs at the sale price?

4. Unit cost

a. Mrs. Brown went to the Confusers Supermarket to purchase a bottle of salad oil. On the shelves she found various shapes and sizes of bottles labeled as follows:

"Regular Size"@ 32¢ ; "Colossal Size" @ 59¢
"Giant Size" @ 45¢ ; "Super Colossal" @ $1.25

What suggestions could you make to help Mrs. Brown make the best price selection from the above?

Elicit from pupils:

1) Mrs. Brown's personal needs. (i.e., small size for a small family prevents waste, large size for large family if economical in price, etc.) Twice the size in one package should cost less than 2 small packages, etc.

2) Mrs. Brown should:

   a) read the label on each size to learn the quantity of oil in each of the above
   b) find the unit cost in each case
   c) use her knowledge of arithmetic to determine which is the best buy.

b. Development

1) A bottle of salad oil labeled "Regular Size" is priced at

32¢. The label states that the contents are 1 quart. What is the cost of the salad oil per liquid ounce?

2) The label on the "Giant Size" states that the contents are 1 quart and 8 ounces and costs 45¢. How does this size compare as a possible purchase with the "Regular Size"? Explain.

3) Since Mrs. Brown has a large family and prepares many dishes using salad oil, she examines the "Colossal Size" containing one-half gallon and selling at 59¢. What are the contents of this size in quarts? Pints? Ounces? Should Mrs. Brown purchase this size? Explain your answer.

4) You are the manager of a small luncheonette. You sell many dishes using salad oil. The "Super Colossal Size" label states that the contents are one gallon. Should you buy this size? Explain your answer.

c. Problems

1) Two brands of tuna fish are offered for sale. Brand "A" sells at 35¢ per can of 7 ounces, and Brand "B" sells at 32¢ per can of 6 ounces. Both brands are labeled as containing the same quality of fish. Which brand would you select? Explain your decision.

*2) The "XYZ" Stores advertised special offers of their one pound loaf of bread at 2 loaves for 41¢, and their "Super Size" loaf weighing 1 pound, 8 ounces for 35¢. In your opinion, which offer is the better buy? Explain and prove that your decision is the correct one.

3) In my neighborhood, milk has been selling for 25¢ per quart. Yesterday, the price was increased to 26¢ per quart. When I mentioned this to my neighbor, he remarked, "It's only a penny." This bothered me, so I sat down and tried to get the answers to the following questions:

   a) We are a large family using four quarts of milk per day. How many quarts of milk are consumed in a week of 7 days? How many quarts of milk are consumed in one year?

   b) What is the additional cost of our milk per week? Per year?

   c) My salary will not be increased this year. How will the additional cost be met?

   d) What was the per cent of increase in the price of milk when the price was advanced from 25¢ to 26¢?

B. Checking sales slips:

1. New York City Sales Tax

THE GENERAL STORE
246 Sands Street
Brooklyn, N.Y., 10005

Tel. SA 2-4506

6/21/64

|  |  |
|---|---|
| 3 | Shirts @ $3.98 ea. |
| 2 | Ties @ $1.69 ea. |
| 6 | Handkerchiefs @ $ .49 ea. |

Item Total

Sales Tax (4%)

Total

a. Enter the correct amount for each item above.

b. What is the total amount for the items above?

c. What is the Sales Tax for the above bill? (See Sales Tax Chart: III-C-2)

d. What is the total cost including the Sales Tax?

2. Federal Tax

RIVERA SPECIALTIES SHOP
496 South Street
Bronx, N.Y., 10057

Tel. 123-7654

7/20/64

| | | | | |
|---|---|---|---|---|
| 1 | Blouse | @ $5.98 | | |
| 3 | pairs Hosiery | @ 3/$1.98 | | |
| 1 | Pocketbook | @ $4.98 | | |
| | | Item Total | | |
| | | Federal Tax (10%) | | |
| | | Sales Tax (4%) | | |
| | | Total | | |

a. Complete and find the total cost of the items listed in the bill above.

b. At 10%, what is the Federal Tax on the pocketbook?

c. Find the Sales Tax on the cost of the three items. (See Sales Tax Chart: III-C-2)

d. What is the total cost of the items including all taxes?

e. May we apply the 10% Federal Tax to the other items? Explain.

3. Checking sales slips with discount

WHOLESALERS, INC.
265 Front Street
New York, N.Y., 10007

Tel. 123-7654

7/20/64

| | | | | |
|---|---|---|---|---|
| 36 | Rope Coils | @ $2.78 ea. | 99 | 08 |
| 24 | Ball Pins | @ $4.69 ea. | 112 | 56 |
| 18 | Carts | @ $3.75 ea. | 67 | 50 |
| | | Item Total | 289 | 14 |

a. The above bill was sent to Mary's employer, who in turn, asked her to check it.

What steps would you take if you held Mary's job? Explain.

b. What errors, if any, were made in this bill?

c. Make any necessary corrections and rewrite the bill.

d. If the seller allows a 2% discount for prompt payment, how much could you deduct from the bill? What would be the correct amount to pay after deducting the discount?

4. Finding and correcting errors

ROME MUSIC SHOP
426 Pizza Road
Brooklyn, N.Y., 11226

Tel. EL 5-7890

6/10/64

| | | | |
|---|---|---|---|
| 3 | Albums @ $2.98 ea. | 6 | 94 |
| 1 | Record Player #1290 | 32 | 50 |
| 6 | packages of Needles @ $ .59 ea. | 3 | 54 |
| | Item Total | 31 | 98 |
| | Sales Tax | 1 | 28 |
| | Total | 32 | 16 |

A customer bought the above items from one of the salesclerks who wrote the above bill.

As the cashier, you are responsible for all mistakes.

a. What, if any, errors has the salesclerk committed?

b. Rewrite the bill, entering any corrected amounts for the items on the bill.

c. Find the correct total.

d. Could you explain the clerk's mistakes to him, and so help him for the future?

### C. Making change

Marie is a cashier in a bakery shop. She has been told to use as small a number of coins as possible in making change. Why?

During the day, she had to complete the following transactions. What change would you expect if you were the purchaser?

| Amount Presented by Customer | Amount of Purchase | Pennies | Nickels | Dimes | Quarters | Half Dollars | Dollar Bills |
|---|---|---|---|---|---|---|---|
| $5.00 | $2.64 | 1 | 0 | 1 | 1 | 0 | 2 |
| $1.00 | $ .32 | | | | | | |
| $10.00 | $6.47 | | | | | | |
| $5.00 | $1.11 | | | | | | |
| $1.00 | $ .53 | | | | | | |
| $10.00 | $5.34 | | | | | | |

### D. Trading stamps

Have the pupils recall that many stores issue trading stamps on most items purchased. Based on the usual practice, the customer receives one stamp for each 10¢ spent.

1. How many trading stamps should a customer receive for a purchase of $ .60?, $2.80?, $5.40?, $13.60?
2. The pupils should understand that since parts of a stamp cannot be issued, the purchase price is rounded off in the following manner:

   $1.72 will receive 17 stamps; $3.58 will receive 35 stamps; and $2.35 will receive 23 stamps.
3. Have pupils complete the following table:

| Purchase Price | Number of Stamps |
|---|---|
| $4.50 | |
| $12.72 | |
| $6.89 | |
| $10.06 | |
| $15.21 | |

## V. Banking

### A. Savings banks

1. When Joe received his first paycheck, he told his friend that he intended to save part of his pay each week. He said he intended to put his savings in a metal box and keep it in his closet .

    a. Was Joe's idea of saving money each week a good one?

    b. Is his idea of saving money in the metal box a good one? Why?

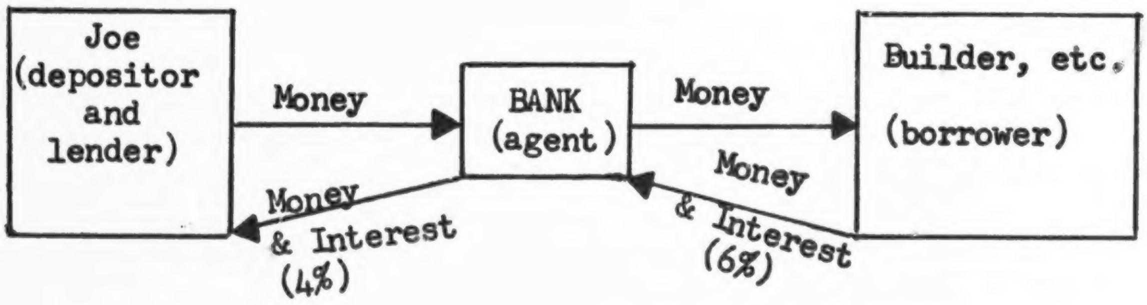
    c. Where would you advise Joe to keep his savings? Why? (safer, less likely to spend it, he earns interest)

    d. What does the bank do with the money Joe and other people deposit? Why?

    e. What services does the bank give to its depositors?

    f. What proof does Joe get from the bank to show that he deposited his money?

2. a. Explain the diagram below:



    b. What does the bank do with the 2% difference in interest?

3. a. When Joe deposited his money in his savings account, he had to make out a deposit slip. Assume he planned to deposit $5.16 in cash and two checks, a birthday gift check for $10.00, and another for $3.64 which he received for helping a neighbor. Make out Joe's deposit slip.

XYZ SAVINGS BANK
246 Fulton Street
Brooklyn, N.Y., 11226

Acct. #____ Date:______

Name________________________

Address______________________

| | | |
|---|---|---|
| 1. Cash ______________ | | |
| 2. Checks ____________ | | |
| 3. " ______________ | | |
| 4. " ______________ | | |
| Total ______________ | | |

b. Before Joe deposited this money, he showed you his bank book. The last figure was $278.63. What should be the figure in the bank book after he makes this deposit?

c. Assume the quarterly interest rate of 1%. How much interest will the bank pay Joe three months later? (assume no additional deposits)

d. Suggested practice:

Use either rexographed or printed forms (G-1 list) and have pupils practice making out deposit slips.

B. Checking accounts

1. Opening a checking account

a. Mr. Coleman ordered baseball uniforms from the ABC Sporting Goods Company in Chicago. When the uniforms arrived, the enclosed bill was for $215.50.

Since Mr. Coleman must teach his classes at school and cannot go to Chicago, Al, the team mascot, offered to go to Chicago to pay the bill provided the school would pay his fare of $80.15.

Is this a practical way to get the bill paid?

What would you suggest? (by check)

Elicit from pupils the advantages of sending a check.

Elicit from pupils that Mr. Coleman had to have a checking account at the bank before he could write a check.

b. Develop some of the steps Mr. Coleman had to take in opening a checking account.

A personal information card
Signature card
Business or professional reference card
Must make a minimum deposit
The bank gave him a check book

2. Writing checks:

a. Help the pupil fill in the check and the stub for the above transaction.

Assume that Mr. Coleman had on deposit $512.00.
(Use either rexographed forms or printed forms from G-1 List)

#201 Amount_____
________19___
To______________
For_____________

| | Dol. | Cents |
|---|---|---|
| Bal. Frwd. | | |
| Deposits | | |
| Total | | |
| This check | | |
| Bal.Frwd. | | |

#201 New York, N.Y. ____19___
The Creditors National Bank
Pay to the order of
____________________________________ $
____________________________________ Dollars
____________________

1) What is the purpose of the stub?
2) What facts must the check show?
3) Why is the amount of the check written twice?
4) Why must the space be filled in between the written amount and the printed word, "Dollars"?
5) With what must the form of Mr. Coleman's signature agree? (Explain to pupils that Mr. Coleman has signed the signature card on file at the bank as Joseph B. Coleman. He cannot use J. Coleman in signing his checks.)

Caution the pupils that no erasures or changes are permitted. If an error is made, the check must be rewritten.

Elicit from the pupils why the bank makes this requirement.

3. Endorsements:

a. Tell the pupils that the ABC Sporting Goods Company is called the "payee", because they are the ones who are getting paid.

   Mr. Coleman is called the "payer" because he is the one who signed the check, and is paying.

b. Have each pupil separate his check from the stub and exchange checks with his neighbors. Each pupil now plays the role of the "payee". Why?

c. Develop with the pupil the meaning and importance of the endorsement.

d. Have pupils endorse the check.

   Note: Where the teacher feels the pupils may profit from a discussion of the various types of endorsement: blank, limited, and full, the topic may be treated here.

4. Collecting on a check

a. What steps must be taken by the ABC Sporting Goods Company after the endorsement in order to collect the $215.50?

b. What happens at Mr. Coleman's bank when his check is presented for collection?

c. What is Mr. Coleman's balance after the check is charged against his account?

d. What does Mr. Coleman's bank do with his check:

e. Why should Mr. Coleman save his checks?

   Note: In some classes, the teacher may wish to explain briefly the work done by the Clearing House. (Materials available from the Federal Reserve Bank for this treatment.)

5. Practice

Distribute to class a rexographed sheet or printed sheet showing three stubs and checks.

Present the following problem:

Mr. Morris has a balance on April 1 of $385.25. He uses checks to pay the following bills.

| Date | Bill | Amount |
|---|---|---|
| 4/2 | New York Telephone Company | $ 8.56 |
| 4/3 | R. H. Macy Company | 125.03 |
| 4/5 | Star Furniture Company | 40.15 |

On 4/3, he made a deposit of $35.00.

Fill in the stubs and checks and compute the final balance.

C. Let the Buyer Beware (Caveat Emptor)

1. The cost of money

Note: Pupils should be aware of the difference between the payment for a ticket to a movie where the price is known and the payment for borrowing money (loan, installment credit), where the true cost is often disguised. The true interest rate is usually higher than the purchaser is lead to believe. Computation of the true rate of interest is optional. A detailed treatment of this topic may be found in various texts. The purpose of the discussion below is to alert the pupil to the cost of consumer credit.

a. Present the problem

A small loan company quotes an interest rate of 1½-2½ per cent per month.

1) What is the rate of interest per year? At 1½%? At 2½%?

2) Why does the company quote the rate by the month when most loans are made by the year?

b. Mr. Smith, owner of the Smith Hardware Company, wished to borrow $300. Because he was an established business man, the Eighth National Bank gave him a commercial loan for one year at 6%.

1) What was the interest on Mr. Smith's loan?
2) What was the total amount Mr. Smith paid back to the bank?
3) Why does the bank give a special rate of interest to an established business man? (good risk)
4) What would be the interest if Mr. Smith took the loan for only 90 days?

c. An advertisement in the newspaper offers to lend $300 on the borrower's signature. According to the advertisement, the borrower is to repay the loan in 12 equal monthly payments of $28.

1) What does the phrase, "on the borrower's signature" mean?
2) To what type of borrower was this advertisement directed? (poor risk, i.e., seasonal worker, part time worker, no bank credit)
3) How much must be paid back to the loan company for this $300 loan?
4) What rate of interest is the company charging? (approximate)

2. Installment buying

a. Present the problem

Mr. Brown wishes to purchase a washer-dryer combination which is listed for $650. The store will allow a 10% discount for a cash purchase, or the machine can be bought on the "easy payment plan" for $50 down and $30 a month for 24 months.

1) What is the cash purchase price of the machine?
2) What would be the total cost to Mr. Brown if he used the installment plan?
3) What is the difference between the cash purchase price and the installment purchase price?

b. Present the problem

A friend suggested to Mr. Brown that he borrow money from the South River Savings Bank and purchase the machine for the cash discount price. The bank will lend Mr. Brown $600 for which he is to make 24 monthly payments of $27 each.

1) What would be the total cost to Mr. Brown of this loan?
2) What is the difference between this and the cash discount price?
3) What is the advantage to Mr. Brown of borrowing the money from the bank rather than buying on the installment plan?

# V.

# STATISTICS

## I. Reading and Interpreting Statistical Tables

### A. Suggested procedure

1. Have pupil realize the difficulty of interpreting data which has not been organized.

    a. Present a situation such as: 5 members were to be selected for a Student Court in Thomas Paine Junior High School. The voting showed the following results: Al, 268; Anna, 119; Ben, 201; Bob, 278; Fred, 312; Grace, 134; Harry, 165; Inez, 302; John, 281; Lydia, 268; Marie, 178; Sam, 149.

    b. Have pupil answer such questions as:

        1) Who was elected?
        2) How many votes did each winner have?
        3) How many votes were cast?
        4) Did Fred receive a majority of the votes?
        5) How is information of this type usually recorded?

2. Have pupil suggest organizing the information in I-A-1-a in table form.

Voting Results for Student Court

| Names | Number of Votes |
|---|---|
| Al | 268 |
| Anna | 119 |
| Ben | 201 |
| Bob | 278 |
| Fred | 312 |
| Grace | 134 |
| Harry | 165 |
| Inez | 302 |
| John | 281 |
| Lydia | 268 |
| Marie | 178 |
| Sam | 149 |

Note: Have pupil suggest another way of organizing the table (according to the number of votes).

3. Have pupil realize the organization of information in table form lends itself to easy interpretation.

   Have him answer the question in I-A-1-b. The pupil will notice how much more readily the answers can be found.

B. Suggested practice in reading tables

1.

Record of Temperatures in New York on May 22, 1960

| | Temp. | Hu. | | Temp. | Hu. |
|---|---|---|---|---|---|
| Midnight | 58 | 90 | Noon | 63 | 74 |
| 1 a.m. | 58 | 93 | 1 p.m. | 65 | 68 |
| 2 a.m. | 56 | 95 | 2 p.m. | 66 | 62 |
| 3 a.m. | 57 | 96 | 3 p.m. | 68 | 59 |
| 4 a.m. | 58 | 91 | 4 p.m. | 68 | 53 |
| 5 a.m. | 57 | 94 | 5 p.m. | 65 | 85 |
| 6 a.m. | 56 | 96 | 6 p.m. | 63 | 85 |
| 7 a.m. | 57 | 94 | 7 p.m. | 60 | 80 |
| 8 a.m. | 56 | 97 | 8 p.m. | 61 | 75 |
| 9 a.m. | 58 | 89 | 9 p.m. | 60 | 75 |
| 10 a.m. | 61 | 81 | 10 p.m. | 59 | 75 |
| 11 a.m. | 60 | 83 | 11 p.m. | 59 | 76 |
| | | | Midnight | 59 | 77 |

a. What was the highest temperature of the day?
b. At what time did it occur?
c. What was the lowest temperature of the day?
d. At what time did it occur?
e. What was the difference between the high and the low?
f. How did the temperature change from midnight to 8 a.m.? From 9 a.m. to 6 p.m.? From 7 p.m. to Midnight?
g. The normal average temperature for this date was 63°. How much higher was the highest temperature on this day? How much lower was the lowest temperature?
h. The highest temperature for this date was 91.6° in 1931.
i. How much lower was the highest temperature on May 22, 1960?
j. The lowest temperature for this date was 43.0 in 1925. How much higher was the lowest temperature on May 22, 1960?
k. What was the difference between the highest and lowest humidity for this day?

2.

Comparison of Sales in Department Stores 1963-1964

Four Week Period Ending May 14, 1963

| Federal Reserve Districts | Per cent of Change (Since last report) |
|---|---|
| Boston | +1 |
| New York | +9 |
| Philadelphia | +3 |
| Cleveland | +4 |
| Richmond | -1 |
| Atlanta | +3 |
| Chicago | +1 |
| St. Louis | 0 |
| Minneapolis | +4 |
| Kansas City | -4 |
| Dallas | -7 |
| San Francisco | 0 |

a. Would you say there had been a general increase or general decrease in sales? Explain.
b. Which cities did better this year than last year?
c. Which city made the poorest showing?
d. What happened in the San Francisco and St. Louis areas?
e. What was the per cent of increase in New York over last report?

3. Other tables may be found in the World Almanac and in the New York Times as well as in other sources. Ask similar questions.

II. Making Decisions Based on Information from Tables

A. Suggested Procedure

1. Have pupil interpret information from table and then reach decision based on this interpretation.

Present situations such as:

a. Tryouts were held for the school bowling team on Monday, Tuesday and Wednesday one week. The best daily scores of the ten boys who appeared for the tryouts are as follows:

**Best Daily Scores of Bowling Group**

| Names | Monday | Tuesday | Wednesday |
|---|---|---|---|
| Leonard | 145 | 127 | 148 |
| Dick | 128 | 114 | 137 |
| Harry | 148 | 151 | 132 |
| Peter | 139 | 128 | 115 |
| Bob | 168 | 141 | 149 |
| Ira | 123 | 143 | 120 |
| George | 144 | 126 | 147 |
| Irving | 127 | 113 | 136 |
| Sam | 138 | 150 | 131 |
| Gordon | 121 | 127 | 114 |

Have pupil answer such questions as:

1) Which was the best bowling score on Monday? on Tuesday? on Wednesday?

2) What place did the boy who was in first place on Monday occupy on Tuesday? on Wednesday?

3) Which boy showed the greatest variation in scores? The smallest variation in scores?

4) Based on their scores, which boys should be on the first team of five? Which two should be the substitute?

b. Two Yankee fans were discussing American League Parks. By using this table from The World Almanac, can you answer their questions below?

**Home Run Distances in Some American League Ball Parks**

| City | Park | Feet from Plate to Fence<br>LF | <br>CF | <br>RF |
|---|---|---|---|---|
| New York | Yankee Stadium | 301 | 461 | 296 |
| Boston | Fenway Park | 315 | 420 | 302 |
| Cleveland | Municipal Stadium | 320 | 410 | 320 |
| Detroit | Briggs Stadium | 340 | 440 | 325 |
| Chicago | Comiskey Park | 352 | 415 | 352 |
| Washington | Griffith Stadium | 350 | 401 | 320 |
| Baltimore | Memorial Stadium | 309 | 425 | 309 |
| Kansas City | Kansas City Stadium | 330 | 421 | 353 |

Have pupils answer such questions as:

1) In which parks is the distance from home plate to fence in left field greater than that in right field?

2) Is there any park in which the distance from home plate to the fence in right field is greater than that to the one in left field? Which?

3) In which park is the distance from home plate to center field the greatest?

4) How much greater is that than the next longest distance? (see #3)

5) Babe Ruth hit the longest home run on record in 1919. The ball traveled 587'. In which park could this be a home run?

B. Suggested practice

1. Suits available at a sale:

| | LOOK FOR YOUR SUIT SIZE HERE | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Sizes | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 46 | 48 | 50 |
| Regular | 26 | 43 | 67 | 160 | 253 | 375 | 105 | 395 | 59 | 98 | 56 | 8 | 1 |
| Short | 4 | 15 | 37 | 50 | 46 | 61 | 27 | 46 | 9 | 5 | 3 | | |
| Long | | 3 | 10 | 29 | 67 | 92 | 111 | 27 | 26 | 46 | 11 | 4 | 1 |
| Medium long | | | 3 | 8 | 26 | 61 | 37 | 36 | 27 | 9 | | | |
| Extra short | | | 5 | 4 | 8 | 1 | 2 | 3 | | | | | |
| Extra long | | | | 4 | 8 | 14 | 1 | 22 | 5 | | 13 | 10 | 3 |
| Portly | | | | | 3 | 14 | 3 | 9 | 8 | 7 | 2 | 1 | |
| Short portly | | | | 3 | 3 | 10 | 5 | 8 | 5 | 5 | 2 | | |
| Slenderizer | | | | | | 1 | | 3 | | 1 | 3 | 1 | |
| Short slenderizer | | | | | 1 | 1 | 1 | 3 | 2 | 5 | 1 | | |
| Long slenderizer | | | | | | | 3 | 5 | 5 | 1 | 2 | | |

Note: Much of the illustrative material in this chapter and similar text materials lend themselves to use in the opaque projector or, by reproduction as transparencies, for use in the overhead projector.

Have pupil answer such questions as:

a. How many suits size 36 are available?

b. How many of the size 36 suits are also regular? short? long?

c. A man needs a size 44 extra short. Can he take advantage of this sale? Explain.

d. In which size is there the greatest number of suits?

*2. Amortization schedule - 5%, 12 years, $1,000
Payment - $9.25 a month

| Payment | Amount for Interest | Applied on Principal | Balance Due on Principal |
|---|---|---|---|
| 1 | $4.17 | $5.08 | $994.92 |
| 2 | 4.14 | 5.11 | 989.81 |
| 3 | 4.12 | 5.13 | 984.68 |
| 4 | 4.10 | 5.15 | 979.53 |
| 5 | 4.08 | 5.17 | 974.36 |
| 6 | 4.06 | 5.19 | 969.17 |
| 7 | 4.04 | 5.21 | 963.96 |
| 8 | 4.02 | 5.23 | 958.73 |
| 9 | 3.99 | 5.26 | 953.47 |
| 10 | 3.97 | 5.28 | 948.19 |
| 11 | 3.95 | 5.30 | 942.89 |
| 12 | 3.93 | 5.32 | 937.57 |

*Note: Amortization is the reduction of a debt gradually by regular payments.

Have pupil answer such questions as:

a. What is the amount of the loan?
b. What is the length of time the loan will last?
c. What is the rate of interest?
d. How much will be paid monthly?
e. How much of this is the first interest payment?
f. In the column marked, "Applied on Principal", the first amount is $5.08. Why? How can you check this?
g. What is the balance still due after the first payment?
h. What is the balance still due after the second payment?
i. How does the amount in the column, "Amount for Interest" change from month to month?
j. How does the amount in the column, "Applied on Principal" change as the interest decreases?
k. How much was still due on the loan after the twelfth payment?
l. How much was paid off on the loan when the twelfth payment was made?
m. How many payments will be made in 12 years?

3. Premiums per $1000 for 20-year Endowment Policy

| Age | 20-year Endowment | Age | 20-year Endowment |
|---|---|---|---|
| 20 | $48.00 | 45 | $54.51 |
| 25 | 48.46 | 50 | 58.96 |
| 30 | 49.12 | 55 | 66.47 |
| 35 | 50.11 | 60 | 78.55 |

Have pupil answer such questions as:

a. How much more for each $1000 of insurance is the annual premium for a man of 50 than for a man of 25? Why do the companies charge more?

b. Mr. Jones, 30 years old, takes a $5000, 20-year endowment policy. The premium per $1000 is $49.12. What is his annual premium?

c. If Mr. Clark takes a 20-year endowment policy for $1000 when he is 25 years old, how much will he pay on premiums in the 20 years?

MAJOR LEAGUE BASEBALL
Tuesday, July 7, 1964

4.

American League
Won and Lost Record

| | Baltimore | Chicago | New York | Minnesota | Detroit | Boston | Los Angeles | Cleveland | Kansas City | Washington | Won |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Baltimore ... | -- | 5 | 5 | 5 | 5 | 7 | 7 | 1 | 7 | 6 | 48 |
| Chicago ..... | 5 | -- | 0 | 3 | 7 | 7 | 4 | 8 | 3 | 7 | 44 |
| New York .... | 3 | 10 | — | 4 | 3 | 4 | 6 | 6 | 5 | 4 | 45 |
| Minnesota ... | 4 | 5 | 6 | — | 4 | 6 | 4 | 4 | 4 | 6 | 43 |
| Detroit...... | 2 | 1 | 4 | 8 | — | 2 | 5 | 5 | 5 | 6 | 38 |
| Boston ...... | 5 | 1 | 6 | 3 | 2 | — | 6 | 4 | 7 | 4 | 38 |
| Los Angeles . | 2 | 2 | 3 | 2 | 2 | 4 | — | 8 | 6 | 8 | 37 |
| Cleveland ... | 4 | 3 | 0 | 6 | 4 | 5 | 4 | — | 3 | 5 | 34 |
| Kansas City.. | 1 | 1 | 4 | 3 | 6 | 3 | 5 | 3 | — | 5 | 31 |
| Washington .. | 2 | 2 | 3 | 3 | 6 | 3 | 3 | 3 | 7 | — | 32 |
| Lost ....... | 28 | 30 | 31 | 37 | 39 | 41 | 44 | 42 | 47 | 51 | — |

**National League**
**Won and Lost Record**

| | Philadelphia | San Francisco | Cincinnati | Pittsburgh | St. Louis | Los Angeles | Milwaukee | Chicago | Houston | New York | Won |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Philadelphia .. | — | 4 | 4 | 4 | 3 | 5 | 3 | 8 | 7 | 9 | 47 |
| San Francisco.. | 5 | — | 9 | 4 | 6 | 8 | 4 | 2 | 3 | 6 | 47 |
| Cincinnati .... | 1 | 3 | — | 4 | 6 | 10 | 2 | 8 | 4 | 4 | 42 |
| Pittsburgh .... | 3 | 5 | 4 | — | 3 | 2 | 2 | 4 | 5 | 11 | 40 |
| St. Louis .... | 6 | 6 | 4 | 2 | — | 2 | 6 | 3 | 6 | 4 | 39 |
| Los Angeles.... | 3 | 2 | 2 | 6 | 5 | — | 5 | 3 | 4 | 8 | 38 |
| Milwaukee ..... | 2 | 2 | 3 | 4 | 6 | 7 | — | 4 | 4 | 6 | 38 |
| Chicago........ | 4 | 1 | 4 | 5 | 3 | 1 | 6 | — | 7 | 5 | 36 |
| Houston........ | 2 | 3 | 3 | 4 | 6 | 2 | 9 | 3 | — | 5 | 37 |
| New York....... | 2 | 5 | 2 | 2 | 2 | 1 | 3 | 3 | 3 | — | 23 |
| Lost .......... | 28 | 31 | 35 | 35 | 40 | 39 | 40 | 38 | 43 | 58 | |

Note: In order to determine the number of games won by any team from the other teams in the league, it is necessary to read: across the row next to the team name.

In order to determine the number of games lost by any team to the other teams in the league, read: down the column under team name.

Have pupil answer such questions as:

a. How many times did Detroit (A.L.) defeat Chicago (A.L.)?
b. How many times did Detroit (A.L.) lose to Chicago (A.L.)?
c. How can you check the total number of games played by each team?
d. Which team in the N.L. played the greatest number of games to this date?
e. Which team in the N.L. played the fewest number of games to this date?
f. Which team seemed to be the most difficult for Los Angeles (N.L.) to defeat?
g. Which team had defeated Milwaukee the greatest number of times?
h. Have pupil bring in a "Standing of the Clubs" chart from newspaper (September - near end of season). Did the trends that seemed to be developing continue throughout the season?

## III. Constructing Statistical Tables

### A. Suggested Procedure

1. Have pupil review need for organizing information in table form (see I-A-2)

2. Have pupil realize the importance of the title of a table and suitable headings of columns.

   Present a situation such as:

   Last March, in our Mathematics and Science Fair, projects were entered by pupils. Class 7-1 entered 2 mathematics exhibits; 7-2 entered 5 mathematics exhibits and 2 science exhibits; 7-3 entered 2 mathematics exhibits and 0 science exhibits; and so on for each class.

   a. The principal wants to know how many pupils in each class submitted work. Have pupil construct table which will give this information. A table such as the following should be constructed:

SCIENCE FAIR EXHIBITS BY CLASSES

| | Exhibits | | |
|---|---|---|---|
| Classes | Mathematics | Science | Total |
| 7-1 | 2 | 3 | |
| 7-2 | 5 | 2 | |
| 7-3 | 2 | 0 | |
| 7-4 | 3 | 4 | |
| 7-5 | 3 | 3 | |
| 7-6 | 1 | 2 | |
| 8-1 | 2 | 3 | |
| 8-2 | 1 | 1 | |
| 8-3 | 3 | 5 | |
| 8-4 | 2 | 3 | |
| 8-5 | 2 | 1 | |
| 9-1 | 2 | 4 | |

b. The assistant principal in charge of each grade is interested in the number of pupils in his grade who entered exhibits. Have pupil construct a table which will give him this information. A table such as the following should be constructed:

SCIENCE FAIR EXHIBITS BY CLUBS

| | Exhibits | |
|---|---|---|
| Clubs | Mathematics | Science |
| Mathematics | 19 | 10 |
| Science | 5 | 30 |
| Other Clubs | 5 | 3 |
| Non-club Members | 9 | 9 |

Have pupils answer such questions as:

1) What words in each title tell you how the tables differ from one another?
2) Which table would you use to determine which grade contributed the greatest number of mathematics exhibits? The greatest number of science exhibits?
3) Why is it important to include suitable headings for the columns in each table?

3. Have pupil learn the value of a frequency table to group information and how to construct a frequency table.

   a. Present problem such as: the supplier of caps and gowns for graduation wants to know the heights of the boys and girls in this class so that he can order the right sizes from his manufacturer. In what form shall we send the information?

   b. Have pupils discuss ways of presenting the information on heights: (possible answers: by rows, an alphabetical list of the class; lists of boys and lists of girls; size place list; grouping by sizes).

   c. Help pupils develop tables based on their suggestions. For example,

1) Random order:

**HEIGHTS OF BOYS AND GIRLS IN CLASS 9 - 10**

(as reported)

| Boys | Girls |
|---|---|
| 5'6" | 5'6" |
| 5'6" | 58" |
| 5'1" | 61" |
| 5'4" | 4'7" |
| 5'9" | 4'8" |
| 5'11" | 5'0" |
| 5'6" | 5'2" |
| 5'5" | 68" |
| 5'6" | 5'4" |
| 5'6" | 5'2" |

2) Suggest an arrangement which would be more helpful to the supplier.

4. Ordered data: (size place; size place by sex):

**HEIGHTS OF BOYS AND GIRLS IN CLASS 9-10 IN ORDER OF SIZE**

| Height in Feet and Inches | Inches |
|---|---|
| 4'7" | 55 |
| 4'10" | 58 |
| 5'0" | 60 |

Have pupils realize that repetitions occur and the supplier might find the information more useful if he could see at a glance the number required for each height.

5. Have pupil draw up a table which shows the frequency of each height. Such a table is called a frequency table.

**HEIGHTS OF BOYS AND GIRLS IN CLASS 9-10**

| Height | Tallies | Number required for each height |
|---|---|---|
| 55 | / | 1 |
| 58 | / | 1 |
| 60 | // | 2 |
| 61 | / | 1 |
| 62 | // | 2 |
| 64 | // | 2 |
| 65 | / | 1 |
| 66 | ~~////~~ / | 6 |
| 68 | // | 2 |
| 69 | / | 1 |
| 71 | / | 1 |

Have pupils answer such questions as:

a. How many pupils are 55" tall? 71" tall? 59" tall?
b. How many pupils are more than 61" tall? Less than 58" tall? at least 66" tall?
c. Why is this arrangement more helpful to the supplier than to the other types of arrangements?

B. Suggested Practice

Arrange data in table form:

1. Heights of members of class.
2. Weights of members of class.
3. Number of Patrol members from each grade; each class.
4. Number of absences in school in one week by grade.
5. Number of pupils late this week by grades; by class; by sex.
6. Number of books lost last year: Mathematics, English, Social Studies.
7. Membership in school clubs.
8. G.O. membership by grade; by class.
9. Number of student and non-student tickets sold for school play.
10. Tables based on other school activities

11. Price of same item of food in different stores.

12. Comparison of seating capacities of various auditoriums in the neighborhood for graduation exercises.

13. Tabular list of pupils' shoe sizes; suit sizes; dress sizes.

## IV. Describing Data - Measures of Central Tendencies (averages): Mean, Median and Mode

### A. Suggested Procedure

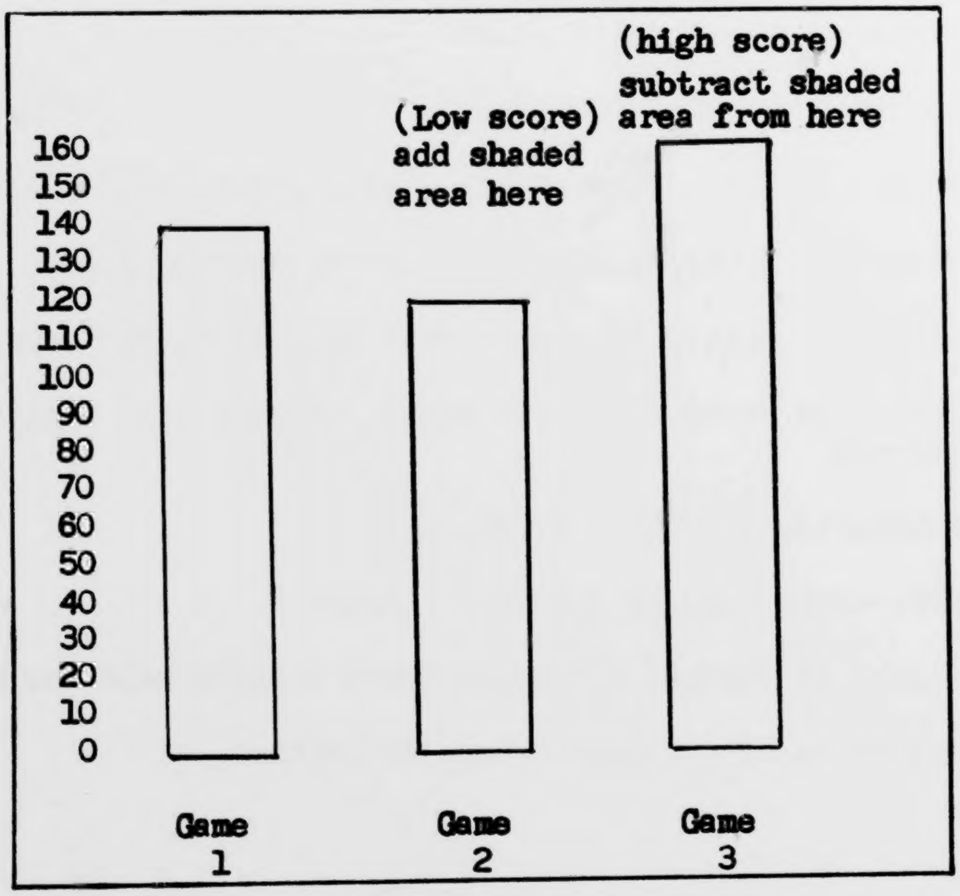
1. Development of Concept of Mean

   a. Have pupil review computation of arithmetic averages:

      1) Pose problem:

John bowls on his high school team. In his last three game series, he had the following scores: 140, 120, 160. What was his series average (mean)?

Have pupil realize that the mean is the score that would give the same total number of pins if all three games had identical scores. He should see that if he subtracts 20 pins from the 160 score and adds it to the 120 score, the three scores would then be 140, 140, 140. He can conclude that the average (mean) is 140.



Using the above technique, find the average of:

| | |
|---|---|
| a) 30, 40, 50 | e) 50, 48, 52 |
| b) 25, 50, 75 | f) 64, 60, 68 |
| c) 55, 65, 60 | g) 98, 72, 85 |
| d) 40, 40, 40 | h) 60, 55, 50, 35 |

2) Pose problem:

In last year's Red Cross drive, Class 9-3 made the following contributions:

10¢, 15¢, 15¢, 15¢, 5¢, 36¢, 13¢, 14¢, 16¢, 20¢, 13¢,
11¢, 10¢, 11¢, 10¢, 10¢, 15¢, 16¢, 20¢, 20¢, 15¢,
10¢, 15¢, 13¢, 10¢, 5¢, 5¢, 16¢, 20¢, 15¢, 15¢.

What is the average contribution?

Solution:

Pupil thinks: Add all the contributions and divide the total by the number of contributors (31). He finds the average is: 14¢.

Help pupil realize that presenting the above information in the form of a frequency table facilitates the computation of the mean:

CONTRIBUTIONS TO THE RED CROSS BY CLASS 9-3

| Amount | Tally | Number of Contributors | Total Contributions |
|---|---|---|---|
| 5¢ | /// | 3 | 15¢ |
| 10¢ | 𝍸 / | 6 | 60¢ |
| 11¢ | // | 2 | 22¢ |
| 13¢ | /// | 3 | 39¢ |
| 14¢ | / | 1 | 14¢ |
| 15¢ | 𝍸 /// | 8 | 120¢ |
| 16¢ | /// | 3 | 48¢ |
| 20¢ | //// | 4 | 80¢ |
| 36¢ | / | 1 | 36¢ |

Average = $\frac{434}{31}$ = 14

The average contribution is 14¢.

Does this mean that each pupil contributed 14¢? Explain.

Guide pupils to an understanding of the meaning of average as a single value* which represents the entire group. It is determined by computing: finding the total of the values and then dividing by the number of addends. *The average contribution 14¢ implies that if all contributions had been equal, each contribution would have been 14¢. Have pupils learn that the word average used in the above description is called the mean.

b. Have pupil practice finding the mean.

c. Pose problems with extreme values and have pupil discover how the average has been altered.

For example, in the problem IV-A-1-a-2) if one of the contributions is $5 instead of 10¢, the average would have been materially affected.

In the first problem the range was 31¢ (36¢-5¢); in this the range is $4.95 ($5 - 5¢). After a number of such problems, guide the pupil to sum up his observations:

1) The value of each score is involved in computing the "average."
2) Extreme values affect the average.

2. Development of the concept of median

a. The meaning:

Choose any seven boys (girls) in the class. Let them arrange themselves in size places, and have each write his height on the board.

1) With reference to their heights, which boy is in the middle? Why?

2) How many boys are below the middle one? How many above?

3) With reference to the numbers on the board, which is the middle number? How do you know?

4) How many numbers are below this middle number? How many above?

This middle position or value is called the median. (The median is the middle point, so chosen in a series that there are as many numbers above it as below it when they are arranged in order of size.)

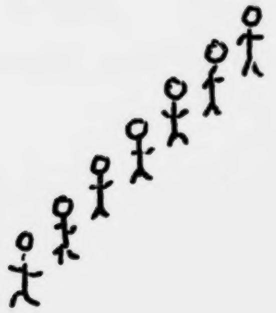
b. Methods of finding the median:

1) Matching

a) An odd number of cases

Have pupil arrange data in order of size either increasing or decreasing.

Have pupil realize that he could find the median by counting.



$$\underbrace{58\ 60\ 61}_{3}\ \Big|\ \overset{\downarrow}{63}\ \Big|\ \underbrace{64\ 65\ 67}_{3}$$

Let pupil pair off the largest value with the smallest value, the next largest with the next smallest, etc. until he has only one left -- the middle value (median).

Have him check that there are as many values above it as below it.

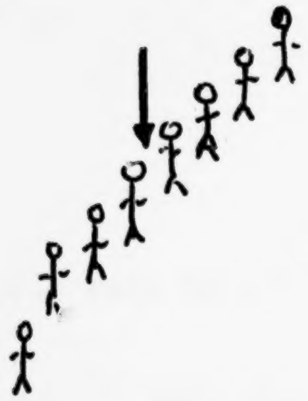
b) An even number of cases

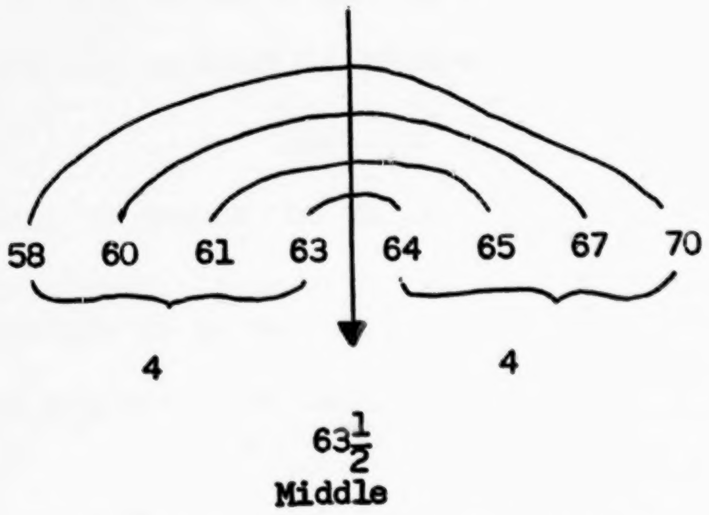
Follow the procedure in a) using eight boys.

Have pupil arrange data in order of size either increasing or decreasing.

Let pupil pair off the largest value with the smallest value, the next largest with the next smallest, etc. -- but now he finishes with a pair. There is no middle value.

Here he needs to find the value halfway between these two.





2) Arithmetic Computation - Arrange the number of cases in order of size

a) An odd number of cases: Subtract one which will represent the middle one and divide the remaining cases into two equal parts. The number obtained tells how many cases are above and below the middle. Let the pupil count this number either up (or down) to the middle value the Median.

67

65

64

(63)

61

60

58

1st step: 7-1 = 6 to be divided

2nd step: $\frac{1}{2}$ of 6 = 3

3rd step: The median is the 4th number.

4th step: The median is 63.

b) An even number of cases: Divide the number of cases into two equal parts. The number obtained tells how many cases are above and below the median. Let the pupil count this number up (or down) to the two middle values. Then he needs to find the value halfway between these middle values. (Take half the difference and add it to the smaller.)

| | | down | |
|---|---|---|---|
| 70 | | 1 | 8 cases |
| 67 | | 2 | $\frac{1}{2}$ of 8 = 4 |
| 65 | | 3 | The median is between the 4th and 5th numbers. |
| 64 | 5 | 4 | The median is between 63 and 64 |
| | The median is $63\frac{1}{2}$ | | |
| 63 | 4 | 5 | |
| 61 | 3 | | |
| 60 | 2 | | |
| 58 | 1 | | |
| | up | | |

c) Have pupil realize:

The median is not influenced by one very large or small value. The median gives the value at which there are an equal number of cases above and below.

d) Have him find the median in IV-A-1-a-2). Have him change the highest contribution to $5 and now find the median. What do you notice?

e) If the heights were 58, 60, 61, 64, 64, 65, 67, 70, what is the median? (64, since the 4th and 5th heights are 64.)

3. Development of concept of mode

a. Pose problem:

Record of the sizes the owner of a dress store kept of dresses sold during one week.

| Size | Number Sold |
|---|---|
| 10 | 4 |
| 12 | 6 |
| 14 | 9 |
| 16 | 15 |
| 18 | 4 |
| 20 | 2 |

Which dress size was sold in greatest quantity?

b. Have pupil learn that the number which occurs most frequently in a list of numbers is called the mode. It is found by inspection and is used when the score or number that occurs most frequently is needed.

c. The mode is a measure of central tendency and it can be found by inspection -- no computation is necessary.

4. Summary of measures of central tendency

a. Pose problem:

Twenty-three ninth grade pupils work after school. The amounts that these pupils earned one week to the nearest half dollar are as follows:

\$2.50, \$8.00, \$6.00, \$4.00, \$6.00, \$5.50, \$4.00, \$9.00, \$5.50, \$8.50, \$4.00, \$4.00, \$5.00, \$8.00, \$10.00, \$9.00, \$7.50, \$8.50, \$4.00, \$3.50, \$8.50, \$5.50, \$4.00.

As part of a vocational guidance project, a committee of pupils collected these facts and was asked to present the statistical findings to the class.

b. Have pupils discuss what statistical data should be reported (number of cases, range, mean, median, mode).

c. Guide pupil to develop the required information as follows:

## AFTER-SCHOOL EARNINGS OF 9th GRADE PUPILS OF JUNIOR HIGH SCHOOL

Week 1

| Earnings | Tally | Number of Pupils |
|---|---|---|
| $10 | / | 1 |
| 9.50 | | 0 |
| 9. | // | 2 |
| 8.50 | /// | 3 |
| 8. | // | 2 |
| 7.50 | / | 1 |
| 7. | | 0 |
| 6.50 | | 0 |
| 6. | // | 2 |
| 5.50 | /// | 3 |
| 5. | / | 1 |
| 4.50 | | 0 |
| 4. | ~~////~~ / | 6 |
| 3.50 | / | 1 |
| 3. | | 0 |
| 2.50 | / | 1 |

d. Have pupil find the following information from the table:

| | | | |
|---|---|---|---|
| Number of cases | 23 | Median (12th value) | $5.50 |
| Range of earnings | $2.50 to $10 | Mean | $6.11 |
| Mode | $4 | | |

e. Have pupil realize that

1) The mean is found by computation
2) The median is found by counting
3) The mode is found by inspection
4) The range may be found by inspection
5) The four measures may differ in any situation.

B. Suggested Practice

1. Have the pupil find the mean, median and mode from data he gathered in III B.

2. Have pupils indicate their choices of occupations. Have them make up a frequency table from this data and find the mode.

3. Recently, it was reported that the median income for American families was $5,300 for the previous year. The average (mean) was $6,520. The mode was $4,600.

a. What income was received most frequently by an American family?

b. How much less is the mode than the mean?

c. What fractional part of the number of American families received more than \$5,300? What part received less than \$5,300?

d. The average (mean) was \$250 higher than that of last year. What was the mean last year?

e. The median income of American families rose to \$5,417 as reported June 20, 1964. How much of an increase is this over the median reported previously?

4. a. Why is the mode the easiest of the measures of central central tendency to find?

b. How do you find the median when you have an odd number of scores?

c. How do you find the median when you have an even number of scores?

d. Why is the mean the most difficult to find?

5. The pupils in classes 7-2 and 8-3 estimated the number of beans in a jar (secret ballot). The true number of beans was 100. They tabulated their responses in a frequency table.

| Number Guessed | Class 7-2 | Class 8-3 |
|---|---|---|
| 0-24 | // | |
| 25-34 | //// | / |
| 35-44 | 𝍸 / | // |
| 45-54 | 𝍸 | 𝍸 𝍸 / |
| 55-64 | 𝍸 | /// |
| 65-74 | / | // |
| 75-84 | /// | // |
| 85-94 | // | / |
| 95-104 | | // |
| 105-114 | | |
| 115-124 | | |
| 125-134 | / | |
| 135-144 | | |
| 145-154 | / | |

a. In what respect is this grouping different from other tables you have made?

b. How many pupils are there in Class 7-2? in Class 8-3?

c. What was the highest estimate in Class 7-2? the lowest? the range?

d. What was the highest estimate in Class 8-3? the lowest? the range?

e. What is the mode in this distribution for Class 7-2? for Class 8-3?

f. What is the median for Class 7-2? for Class 8-3?

g. Which class made better estimates? Explain. (Class 8-3 had the smaller range.)

Note: The Fundamental Concepts in Arithmetic should be developed within the framework of the Statistics Unit.

## FUNDAMENTAL UNDERSTANDINGS RELATED TO THE UNIT ON STATISTICS

### I. Meaning of our decimal system

### A. Place Value

The value of a digit in the number depends on the digit and its place in the number, e.g., 232.

The 2 in hundreds place has a value of 200.

The place value of a digit to the left of another is 10 times the value of the one to its right: (if each digit is the same).

| Hundreds | Tens | Ones |
|---|---|---|
| 3 | 3 | 3 |

e.g., "3" in tens place (30) is 10 times the "3" in ones place (3), and "3" in hundreds place (300) is 10 times the "3" in tens place (30).

The place value of a digit to the right of another is $\frac{1}{10}$ the value of that on the left. The "3" in tens place (30) has a value of $\frac{1}{10}$ the "3" in hundreds place (300).

B. Practice:

1. What is the value of 8 according to its position in each of the following

   498 82 8367 6801

2. a) How many times as large as 10 is 100, 1000?

   b) How many times as large as 20 is 200?

3. What part of 400 is 40? of 90 is 9?

4. Compare the place value represented by the 5's in 1255; 5245; 1525.

C. Period chart

| Thousands | | | Ones | | |
|---|---|---|---|---|---|
| Hundreds | Tens | Ones | Hundreds | Tens | Ones |

1. Reading numbers

   Numbers are grouped in periods of three places: ones, tens, hundreds. The first period is called ones or units; e.g., 325 ones. The second period is called thousands, e.g., 426,000 -- 426 thousands or 426 thousand.

   The numbers in each period are read in groups and each group is given the name of the period, e.g. 785,106 -- 785 thousand, 106 ones.

   We omit the word "ones" in reading that period. 785 thousand, 106.

2. Writing numbers

   The comma is used to separate the periods, to facilitate the heading. Periods are counted from the extreme right to the left, e.g., 31,845.

3. Practice

   a. Place commas correctly: 6431 43523 127956

   b. Read the above numbers.

c. Fill in this cross number puzzle:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | a | b | c | d | ▨ | e | f | g | h |
| | i | | | | ▨ | j | | | |
| k | | | | ▨ | ▨ | ▨ | l | | |
| | m | | | n | ▨ | o | | | |
| | p | | | | ▨ | q | | | |

Across

a - Six thousand, four hundred seventy-three
e - Five thousand, two hundred ninety-six
i - Two thousand, eighty
j - Eight thousand, two
k - One thousand, seven hundred forty
l - Four hundred
m - Nine thousand, two hundred one
o - One thousand, three
p - Three thousand, nine hundred
q - Three thousand, five hundred seventy

Down

a - Sixty-two thousand, seven hundred ninety-three
b - Forty thousand, four hundred twenty-nine
c - Seventy-eight thousand
d - Thirty
e - Fifty-eight
f - Twenty thousand, four hundred five
g - Ninety thousand, seven
h - Sixty-two thousand, thirty
k - One
n - Ten
o - Thirteen

Note: The purpose of this cross-number puzzle is to make practice in writing numbers interesting. It is suggested if more practice is needed in reading and writing numbers that pupils draw their own designs, make up their own cross-number puzzles and have the class work them.

## II. Addition

### A. Decade addition

#### 1. Within decade

$$\begin{array}{r}3\\6\\\hline 9\end{array}\qquad\begin{array}{r}13\\6\\\hline 19\end{array}\qquad\begin{array}{r}23\\6\\\hline 29\end{array}\quad \text{etc.}$$

Pupil should see that the sum of 3 and 6 (or any such combination is constant) and that the decade remains the same, i.e., in adding to tens, the sum remains tens; to twenties, the sum remains twenties.

#### 2. To next decade:

$$\begin{array}{r}6\\4\\\hline 10\end{array}\qquad\begin{array}{r}16\\4\\\hline 20\end{array}\qquad\begin{array}{r}26\\4\\\hline 30\end{array}\quad \text{etc.}$$

The decade in each instance is the next higher one.

This is a preparation for bridging the decade, also combinations such as

$$\begin{array}{r}1\\+9\\\hline\end{array}\qquad\begin{array}{r}2\\+8\\\hline\end{array}\qquad\begin{array}{r}3\\+7\\\hline\end{array}\qquad\begin{array}{r}4\\6\\\hline\end{array}\qquad\begin{array}{r}5\\5\\\hline\end{array}\quad \text{etc. and higher decades.}$$

#### 3. Practice:

1) What is the next decade after 20
after 30
after 40
after 50 etc.

2)

$$\begin{array}{r}35\\5\\\hline\end{array}\qquad\begin{array}{r}49\\1\\\hline\end{array}\qquad\begin{array}{r}82\\8\\\hline\end{array}\qquad\begin{array}{r}63\\7\\\hline\end{array}\qquad\begin{array}{r}47\\3\\\hline\end{array}\qquad\begin{array}{r}52\\8\\\hline\end{array}\quad \text{etc.}$$

#### 4. Bridging the next decade:

$$\begin{array}{r}9\\2\\\hline\end{array}\qquad\begin{array}{r}19\\2\\\hline\end{array}\qquad\begin{array}{r}29\\2\\\hline\end{array}\qquad\begin{array}{r}59\\2\\\hline\end{array}\qquad\begin{array}{r}49\\2\\\hline\end{array}$$

B. Column Addition (adding down or up)

1. Preparation

   a. Counting by 4's starting with any number, e.g., 3 - 3, 7, 11, 15, etc.

   b. Much practice starting with any number and counting by 2, 3, 4, 5, 6, 7, 8, or 9.

2. One column

8 Partial sum, then total sum should be stated: 17, 23 (adding down)
9 or 15, 23, (adding up)
$\underline{6}$

8 11, 20, 26 (adding down)
3 15, 18, 26 (adding up)
9
$\underline{6}$

3. Practice

$$\begin{array}{cccccc} 7 & 8 & 9 & 7 & 5 & 7 \\ 3 & 5 & 4 & 6 & 9 & 3 \\ \underline{7} & \underline{8} & \underline{8} & 6 & 8 & 6 \\ & & & \underline{8} & \underline{9} & \underline{4} \end{array}$$

C. Column Addition Involving Exchange

10 ones can be exchanged for 1 ten
10 tens can be exchanged for 1 hundred.

26 Ones column - sum 12
$\underline{46}$ 12 equals 10 ones and 2 ones
Enter 2 in ones column
Add 1 ten to tens
3, 7
Sum: 72

Adding down
39 Ones column - sum 16, 24
47 Exchange 20 ones for 2 tens
$\underline{78}$ Enter 4 in ones column
Add 2 tens to tens
5, 9, 16
Enter 6 in tens column and 1 hundred (10 tens equal 1 hundred) in hundreds column
Sum: 164

## III. Subtraction Involving Exchange

### A. Subtraction

$$\begin{array}{rr} \text{Tens} & \text{Ones} \\ 8 & 0 \\ -3 & 2 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{rr} \text{Tens} & \text{Ones} \\ 7 & 10 \\ -3 & 2 \\ \hline \end{array}$$

2 ones cannot be subtracted from 0 ones.
Exchange 8 tens for 7 tens and 10 ones.
2 ones from 10 ones leaves 8 ones.
3 tens from 7 tens leaves 4 tens.
Difference 48

$$\begin{array}{rr} \text{Hundreds} & \text{Ones} \\ 7 & 3 \\ -1 & 8 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{rr} \text{Tens} & \text{Ones} \\ 6 & 13 \\ -1 & 8 \\ \hline 5 & 5 \end{array}$$

8 ones cannot be subtracted from 3 ones.
Exchange 7 tens for 6 tens and 10 ones.
10 ones and 3 ones equals 13 ones.
8 ones from 13 ones leaves 5 ones.
1 ten from 6 tens leaves 5 tens.
Difference 55

$$\begin{array}{rrr} \text{Hundreds} & \text{Tens} & \text{Ones} \\ 3 & 0 & 0 \\ -1 & 2 & 8 \\ \hline \end{array}$$

When the minuend is a number having no tens and no ones, it is necessary to make 2 exchanges. The pupil may find it necessary to indicate these exchanges as shown below.

$$\begin{array}{rrr} \text{Hundreds} & \text{Tens} & \text{Ones} \\ 3 & 0 & 0 \\ -1 & 2 & 8 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{rrr} \text{Hundreds} & \text{Tens} & \text{Ones} \\ 2 & 10 & 0 \\ -1 & 2 & 8 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{rrr} \text{Hundreds} & \text{Tens} & \text{Ones} \\ 2 & 9 & 10 \\ -1 & 2 & 8 \\ \hline 1 & 7 & 2 \end{array}$$

B. Practice

Note: It is suggested that practice in addition and subtraction utilize devices such as the following to make the acquisition of these necessary skills palatable.

1. Using the numbers 1 to 9 (each number must be used once and and only once) fill in the little squares in the large square so that the sum of the numbers in each column (3 columns), each row (3 rows) and each diagonal (2 diagonals) is 15. There will be 8 sums of 15 each if you have placed the numbers correctly. Hint: Place the number 5 in the center square.

| | | |
|---|---|---|
| | | |
| | | |
| | | |

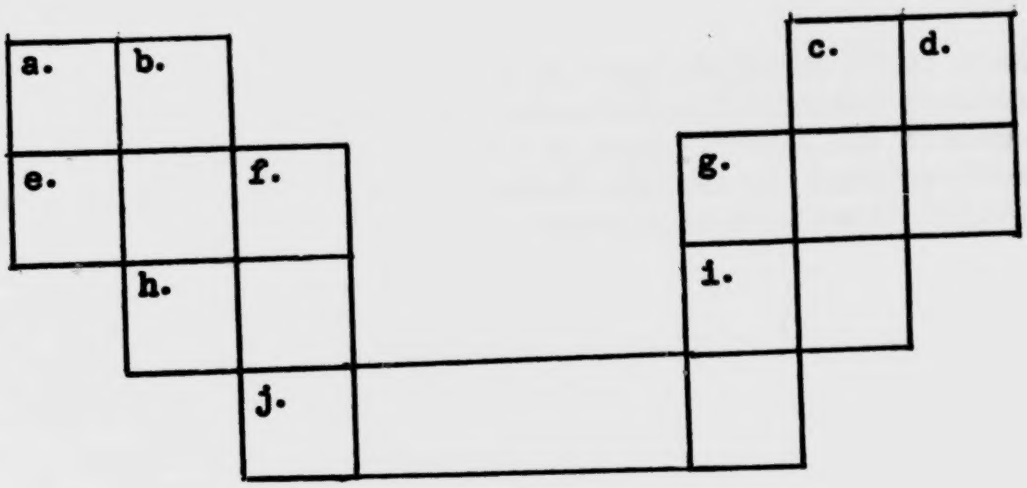
Answer

| | | |
|---|---|---|
| 8 | 1 | 6 |
| 3 | 5 | 7 |
| 4 | 9 | 2 |

2. Try making another magic square using the numbers 2 to 10. Sums: 18

3. Try 3 to 11 Sums: 21
4 to 12 Sums: 24
6 to 14 Sums: 30
7 to 15 Sums: 33
8 to 16 Sums: 36
9 to 17 Sums: 39

4.



**Across**

a. $\begin{array}{r} 39 \\ \underline{+26} \end{array}$

c. $\begin{array}{r} 49 \\ \underline{+28} \end{array}$

e. $\begin{array}{r} 183 \\ \underline{+356} \end{array}$

g. $\begin{array}{r} 849 \\ \underline{+ 93} \end{array}$

h. $\begin{array}{r} 80 \\ \underline{+16} \end{array}$

i. $\begin{array}{r} 30 \\ 36 \\ \underline{9} \end{array}$

j. $\begin{array}{r} 287 \\ 195 \\ \underline{86} \end{array}$

**Down**

a. $\begin{array}{r} 36 \\ 15 \\ 9 \\ \underline{5} \end{array}$

b. $\begin{array}{r} 200 \\ \underline{+ 339} \end{array}$

c. $\begin{array}{r} 1000 \\ \underline{-255} \end{array}$

d. $\begin{array}{r} 2 \\ 55 \\ \underline{15} \end{array}$

f. $\begin{array}{r} 1063 \\ \underline{-98} \end{array}$

g. $\begin{array}{r} 1000 \\ \underline{-22} \end{array}$

## IV. The Role of Zero

**Addition**

6 + 0 $\begin{array}{r} 6 \\ \underline{+0} \end{array}$

0 + 6 $\begin{array}{r} 0 \\ \underline{+6} \end{array}$

0 + 0

**Subtraction**

6 - 0 $\begin{array}{r} 6 \\ \underline{-0} \end{array}$

0 - 0

Have pupil discover that when zero is added to a number the number remains identically the same; that when zero is subtracted from a number, the number remains identically the same. Since adding or subtracting 0 leaves the number identically the same, the zero is called the identity element.

## V. Multiplication

A. Meaning of multiplication as repeated addition of equal addends. 5 x 7 means five 7's or 7 taken 5 times. (7+7+7+7+7).

B. Basic multiplication facts -- reenforcing learnings as developed in K-8.

1. Reenforcement

a. Developing a table through doubling or halving.

| | | | | | |
|---|---|---|---|---|---|
| 1x7 | 5x7 | 2x7 | 4x7 | 3x7 | 3x7 |
| 2x7 | 10x7 | 4x7 | 8x7 | 6x7 | 9x7 |
| | | | | | (tripling) |

b. Arranging in sequence after development:

1x7
2x7
3x7
4x7
5x7
6x7
7x7
8x7
9x7

**Note**: Pupils may be permitted to refer to tables when necessary.

2. The Commutative Law

4 x 7 = 28 $\begin{array}{r}7\\7\\7\\\underline{7}\end{array}$ four 7's = 28

7 x 4 = 28 $\begin{array}{r}4\\4\\4\\4\\4\\4\\\underline{4}\end{array}$ seven 4's = 28

3. Relation through number families. (The division process is the inverse (undoing) of the multiplication process. The division and multiplication facts should be taught simultaneously.)

| | | | |
|---|---|---|---|
| $4 \times 7 = 28$ | $\begin{array}{r}4\\7\overline{)28}\end{array}$ | $7 \times 4 = 28$ | $\begin{array}{r}7\\4\overline{)28}\end{array}$ |
| Four times seven equals twenty-eight | There are four sevens in twenty-eight | Seven times four equals twenty-eight | There are seven fours in twenty-eight |

4. Develop understanding of multiplication tables through counting.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |

Give pupil a rexographed sheet such as above. Have pupil encircle every ninth number. Then have him tabulate his answers.

1 x 9 = 9
2 x 9 = 18
3 x 9 = 27 Finish table

5. Patterns to help in recall.

a. The 9 table:

1 x 9 = 9
2 x 9 = 18
3 x 9 = 27
4 x 9 = 36
5 x 9 = 45
6 x 9 = 54
7 x 9 = 63
8 x 9 = 72
9 x 9 = 81

The sum of the digits of each product is 9, e.g., 1 + 8, 2 + 7, 3 + 6, etc.

Last five products are a mirror reflection of the first five (digits reversed in reverse order)

units digit progresses from 9 — 0
Tens digit progresses from 0 — 9

b. The 2 table

1 x 2 = 2
2 x 2 = 4
3 x 2 = 6
4 x 2 = 8
5 x 2 = 10
etc.

All products are even (2 is an even number. When any number is multiplied by an even number the product is even.)

c. The 4 table

1 x 4 = 4
2 x 4 = 8
3 x 4 = 12
4 x 4 = 16
etc.

All products are even. (4 is an even number so product of 4 and any number is even.)

d. The 5 table

1 x 5 = 5
2 x 5 = 10
3 x 5 = 15
4 x 5 = 20
etc.

The final digit in the product is either 5 or 0. When 5 is multiplied by an even number the product is even, ending in 0.

e. The 10 table. All products end in zero. The tens digit is the same as the multiplier.

f. The 11 table

1 x 11 = 11
2 x 11 = 22
3 x 11 = 33

The product is a number having the multiplier in tens place and in ones place.

6. **Counting**

**To learn 7 table**

**Count by 7's beginning with 7.**
**Count by 7's backward beginning with 70.**

**Same with other tables.**

C. **Multiplication through two places** ("1 place number"is 0-9; "2 place number"is 10-99)

1. **Review counting by a given number starting with any number.**

2. **Multiplication by one place number.**

   a. **A two place number by one place number.**

$$\begin{array}{r} 17 \\ \underline{x4} \end{array}$$

**Have pupil review meaning: of 17 as 10+7, or 7+10. Therefore, 10 is to be multiplied by 4 and 7 is to be multiplied by 4. (Distributive Law)**

$$\begin{array}{rcrcr} 10 & \qquad & 7 & \qquad & 17 \\ \underline{x4} & & \underline{x4} & & \underline{x4} \\ & & & & \overline{28} \\ & & & & \underline{40} \\ & & & & 68 \end{array}$$

**After concept is clear, have pupil use standard algorism.**

$$\begin{array}{cc} \text{Tens} & \text{Ones} \\ 1 & 7 \\ \underline{\phantom{1}} & \underline{4} \end{array}$$

**4 x 7 ones equals 28 ones.**
**Enter 8 in ones place**
**Exchange 20 ones for 2 tens.**
**4 x 1 ten equals 4 tens plus 2 tens (exchanged) is 6 tens**
**Enter 6 in tens place.**
**Product: 68**

b. A 3 place number by a 1 place number

$$\begin{array}{r} 283 \\ \underline{\text{x}7} \end{array}$$

Meaning of 283: 200 + 80 + 3

Therefore 7 x 283 means 7x(200+80+3)

(Distributive Law)

$$\begin{array}{r} 200 \\ \underline{\text{x}7} \\ \phantom{0} \end{array} \qquad \begin{array}{r} 80 \\ \underline{\text{x}7} \\ 560 \end{array} \qquad \begin{array}{r} 3 \\ \underline{\text{x}7} \\ 21 \end{array}$$

Combining:

$$\begin{array}{r} 283 \\ \underline{\text{x}7} \\ 21 \\ 560 \\ \underline{1400} \\ 1981 \end{array}$$

After concept is clear, have pupil use standard algorism.

| | Hundreds | Tens | Ones |
|---|---|---|---|
| | 2 | 8 | 3 |
| | | | 7 |
| 1 | 9 | 8 | 1 |

c. Practice

An interesting arrangement

1 x 8 + 1 = 9

12 x 8 + 2 = 98

123 x 8 + 3 = 987 etc. to 123456789 x 8 + 9 = 987654321

3. Multiplication by a 2 place number

a. Patterns

1) A one place number multiplied by 10 (Review with tens frame if necessary).

10 x 1 = 10
10 x 2 = 20
10 x 3 = 30
10 x 4 = 40
10 x 5 = 50
10 x 6 = 60
10 x 7 = 70
10 x 8 = 80
10 x 9 = 90

In each case zero is in ones place.

2) A one place number multiplied by a multiple of 10, e.g., 20 (2x10)

20 x 1 = 20
20 x 2 = 40
20 x 3 = 60
20 x 4 = 80
20 x 5 = 100
20 x 6 = 120
20 x 7 = 140
20 x 8 = 160
20 x 9 = 180

In each case zero is in ones place. The tens digit is twice the original number.

3) A two place number by 10, e.g.

10 x 11 = 110
10 x 12 = 120
10 x 13 = 130
10 x 14 = 140
10 x 15 = 150
10 x 16 = 160
10 x 17 = 170
10 x 18 = 180
10 x 19 = 190

Have pupil note that the pattern is the same for a one place number multiplied by 10.

b. Multiplying a number by 10

24
x10
equals 24 tens or 240

24
x10
240 Pupil should not multiply each digit by zero.

He should place the zero in ones place (zero is the place holder) and multiply 24 by 1 ten getting 24 tens.

189
x10 189 tens or 1890

c. Multiplying by multiple of 10

43 Enter 0 in ones place
x20 10 x 43 is 43 tens; 20 x 43 is 2 x 43 tens or 86 tens

$$\begin{array}{r} 43 \\ \underline{\times 20} \end{array}$$

Enter 0 in ones place
2 tens times 3 is 6 tens; enter 6
in tens place, and continue multiplying.

d. Multiplying by a 2 place number

$$\begin{array}{r} 27 \\ \underline{\times 35} \end{array}$$

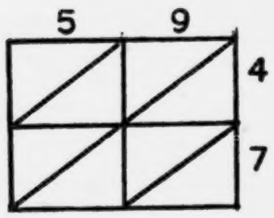
35 equals 30 + 5
Therefore, 35 x 27 is 30 x 27 plus 5 x 27
(Distributive Law)

$$\begin{array}{r} 27 \\ \underline{\times 5} \end{array} \qquad \begin{array}{r} 27 \\ \underline{\times 30} \end{array} \qquad \begin{array}{r} 27 \\ \underline{\times 35} \\ 135 \\ \underline{810} \\ 945 \end{array}$$

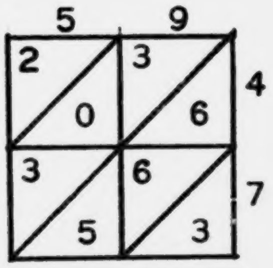
e. Practice

1) Lattice method of multiplication. (This device gives practice in the number facts in multiplication without involvement in exchange. Addition occurs only in the final step.)

47 x 59



Arrange as in diagram.

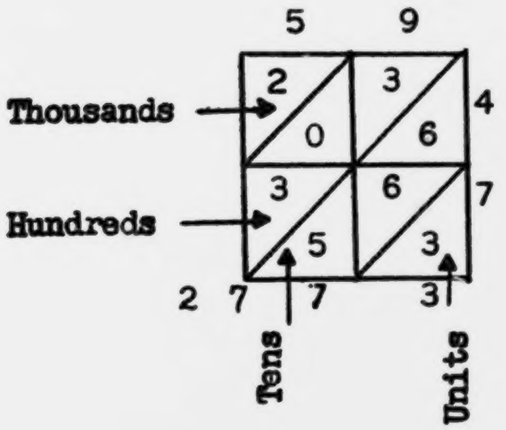


1. Multiply 4 x 9 = 36
   Enter as shown
   Multiply 4 x 5 = 20 and enter as shown.

2. Multiply 7 x 9 = 63
   and 7 x 5 = 35 as shown

3. Add along diagonals, starting at lower right hand corner. Add only the numbers in the triangles

4. Product is 2773



2) Choose your favorite number from 1 through 9. Multiply this number by 9. Now multiply this result by the number 1234567 9. Surprised?

3) The method of duplation (doubling). This method was used for centuries because it required only a knowledge of doubling. 43x29.

| Column I | Column II |
|---|---|
| 43 | 1 |
| 86 | 2 |
| 172 | 4 |
| 344 | 8 |
| 688 | 16 |

Double each number in each column. Continue doubling until a sum of 29 can be obtained in Column II, e.g., 16+8+4+1=29. (omit 2).

Add the numbers opposite 16, 8, 4, 1, - 688+344+172+43 = 1247, the product of 43 and 29.

This same table could be used to multiply 21 by 43, i.e., 43x21.
Column II 16+4+1 = 21
Column I 688+172+43 = 903, the product of 43 and 21

Duplation is an illustration of the fact that a number can be broken up into the sum of a group of numbers, each multiplied by 43 separately and then totaled, e.g., 43 x 21 =

43 x 16 = 688
43 x 4 = 172
43 x 1 = 43
43 x $\overline{21}$ = $\overline{903}$

f. **Multiplication by 50**

50 x 84 -- multiply 84 by 100 and divide by 2. $\frac{8400}{2}$ = 4200

g. **Multiplication by 25**

25 x 33 -- multiply 33 by 100 and divide by 4. $\frac{3300}{4}$ = 825

## VI. Division

### A. Meaning of division as repeated subtraction

1. $13\overline{)\;39}$

$$\begin{array}{r} 39 \\ -13 \\ \hline 26 \\ -13 \\ \hline 13 \\ -13 \\ \hline 0 \end{array}$$

13 can be subtracted from 39 three times or there are 3 13's in 39.

2. $14\overline{)\;72}$

$$\begin{array}{r} 72 \\ -14 \\ \hline 58 \\ -14 \\ \hline 44 \\ -14 \\ \hline 30 \\ -14 \\ \hline 16 \\ -14 \\ \hline 2 \end{array}$$

14 can be subtracted from 72 five times, i.e., there are 5 14's in 72. Two was left undivided. This 2 is the remainder and may be written R 2 or $\frac{2}{14} = \frac{1}{7}$.

### B. Basic number facts

See V-B-3 -- Relationships through number families.
Division is the inverse (undoing of multiplication).

### C. Expressions and symbols indicating division

1. Expressions

Divide 24 by 6.
How many 6's in 24?
6x? is 24.
24 divided by 6.

2. Symbols

$9 \div 4 \qquad 4\overline{)\,9} \qquad \frac{9}{4} \qquad 9:4$

### D. Practice in even division (no remainders)

E. Remainders

1. Practice

a. Remainder of 1: $2\overline{)3}$ $2\overline{)5}$ $2\overline{)7}$ etc.

$2\overline{)9}$ $2\overline{)11}$ $2\overline{)13}$ etc. with all tables

b. Remainder of 2, 3, 4, etc. with appropriate tables.

2. Practice in finding remainders

a. Find

1) Number of yards in a given number of feet, e.g., 4 ft., 5 ft., 6 ft., etc.

2) Number of weeks in a given number of days, e.g., 8 days, 9 days, 10, 11, 12, 13, etc.

3) Number of nickels in a given number of cents, e.g., 21 cents, 22 cents, 23 cents, etc.

4) Number of dimes in a given number of cents or in a given number of nickels, etc.

5) Number of feet in a given number of inches, e.g., 60 in., 61 in., etc.

b. To find whether a number can be divided without a remainder.

1) by 2, test is that it is an even number

2) by 3, e.g., 105 add digits 1 + 0 + 5 = 6. 6 is divisible by 3, so 105 is evenly divisible by 3.

3) by 4, e.g., 124 -- when the last two digits of the number, 24, are divisible by 4, the number is evenly divisible by 4.

4) by 5, any number ending in 0 or 5 is evenly divisible by 5.

5) by 6, any even number the sum of whose digits can be divided by 3 is evenly divisible by 6, e.g., 612 is even and the sum of the digits 6 + 1 + 2 = 9 is divisible by 3.

6) by 9, any number, the sum of whose digits can be divided by 9, is evenly divisible by 9, 3.g., the sum of the digits of the number 729 is 18. 18 is evenly divisible by 9, therefore 729 is evenly divisible by 9

F. Counting - backwards

1. No remainder

by 2's starting with any even number
by 3's starting with any even multiple of 3, e.g., 39

Similarly with 4, 5, 6, 7, 8, 9.

2. Remainders

by 2's starting with an odd number
by 3's so that remainder is 1 or 2

Similarly with 4, 5, 6, 7, 8, 9

G. Division by 2-place divisor

Note to Teacher: A pretest of division by a two place divisor is suggested to determine pupils' specific needs.

Where a review of division by a two place divisor is necessary, the following development, based on successive subtraction is recommended.

1. Review multiplication by 10, 20, 30 ..., also 100, 200 ...

2. How many 15's are there in 360? Estimate: 10 x 15 = 150 (too little)
20 x 15 = 300 (too little)
30 x 15 = 450 (too much)

The answer is somewhere between 20 and 30.

Note: If pupils understand the above method of estimation, proceed to (c) below.

If pupils do not understand the method of estimation, use algorisms as in (a) and/or (b) below, depending on pupils' level of ability.

This method of successive subtraction may be used through the application of any multiplication facts that have been mastered by the pupil.

```
                      24
Computation: (a) 15)360
                    150       10 x 15
                    210
                    150       10 x 15
                     60
                     60        4 x 15
                      0       24 x 15
```

or

```
        24
(b) 15) 360
        300       20 x 15
         60
         60        4 x 15
          0       24
```

or

```
        24
(c) 15) 360
        300       20
         60
         60        4
          0       24
```

What is the meaning of the two in the quotient?
(It shows that 15 has been subtracted 20 times from 360.)

What is the meaning of the four in the quotient?
(It shows that 15 has been subtracted four more times from 360.)

How may we check the answers? (15 x 24 = 360 or 24 x 15 = 360.)

Do your answers check?

3. Problem: $175\overline{)3675}$ Estimate:
10 x 175 = 1750 (too little)
20 x 175 = 3500 (too little)
30 x 175 = 5250 (too much)

The answer lies somewhere between 20 and 30. However, in this instance, it must be closer to 20 since the divided (3675) is much closer to 3500 than to 5250.

**Computation:** (a)

$$\begin{array}{r|r} \multicolumn{1}{r}{21} & \\ 175\overline{)3675} & \\ \underline{1750} & 10 \times 1750 \\ 1925 & \\ \underline{1750} & 10 \times 175 \\ 175 & \\ 175 & 1 \times 175 \\ \hline 0 & 21 \times 175 \end{array}$$

or

(b)

$$\begin{array}{r|r} \multicolumn{1}{r}{21} & \\ 175\overline{)3675} & \\ \underline{3500} & 20 \\ 175 & \\ 175 & 1 \\ \hline 0 & 21 \end{array}$$

Questions: Review as above

What is the meaning of two in the quotient?

What is the meaning of one in the quotient?

How may you check your answers?

# VI

# GRAPHS

## I Statistical Graphs

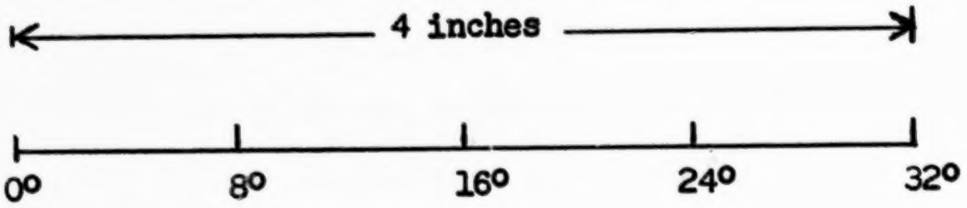
### Suggested Procedure

### A. Scalar Representation

#### 1. Representing a number by a line

If the temperature at noon one day is 32° Fahrenheit, how can we represent this fact by a line? Guide pupil to see that since we cannot conveniently draw a line 32 feet or even 32 inches long, we must choose a much smaller unit. If we choose $\frac{1}{4}$" to represent 1°, we must have a line that is 8" long which is still too long to show conveniently on ordinary paper. $\frac{1}{8}$" to represent 1° would result in a line 4" long, which can easily be drawn on a sheet of paper. Have pupil represent the 32° graphically as a line segment.

Today's Temperature



$\frac{1}{8}$" = 1° is read as $\frac{1}{8}$" represents 1°.

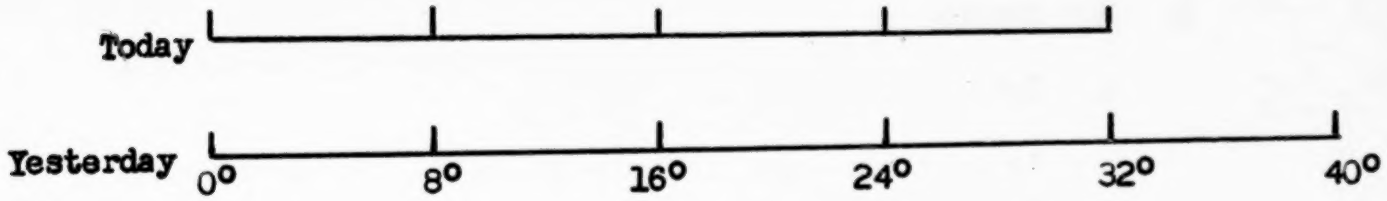
Have pupil practice representing numbers by lengths of lines.

Note: Be sure pupils understand that the quantity of 8° is represented by a one inch length and the point labeled 8° marks the end of the one inch length, etc.

#### 2. Comparing two numbers by lengths of lines.

If today's temperature at noon is 32° and yesterday's was 40°, how can we show the comparison between the two temperatures by means of lines?

Guide pupil to see that we must keep the same unit, that is $\frac{1}{8}$" represents 1°.



3. Choosing a convenient scale

In making a scale drawing of a line 100 yards long, which is the better scale to use: 1" repreents 1 yard, or 1" represents 20 yards? Guide pupil to see that the choice of scale depends on the space available and the numbers involved.

4. Suggested Practice

a. If the scale is $\frac{1}{4}$" to 1', find the numbers needed to fill the blanks in the following table:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Actual Length | 4' | 8' | 12' | 10' | ? | 14' | ? | ? | 6' |
| Scale Length | 1" | 2" | ? | ? | 4" | ? | 5" | $2\frac{1}{2}$" | ? |

b. If the scale is 100 feet to the inch, find the numbers needed in the following table:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Actual Length | 50' | 100' | 200' | ? | 350' | ? | 425' | 575' |
| Scale Length | $\frac{1}{2}$" | 1" | ? | 3" | ? | $1\frac{1}{4}$" | ? | ? |

c. If scale is 1 division on the graph paper to represent 2°, find the numbers needed in the following table:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Actual Length | 4° | 10° | ? | 20° | ? | 15° | 25° | ? |
| Scale Length | 2 | 5 | 8 | 10 | 4 | ? | ? | 15 |

d. Have pupils practice representing some of these scale lengths as lines.

e. Show by line segments:

1) a class of 34 children
2) an honor roll of 18 names
3) the width of the classroom (let pupils measure this)
4) distances from one city to several others

## B. Bar Graphs

### 1. Interpretation

Present the last problem developed in the section on statistics as a vertical bar graph. (Statistics -- p. 141 and 142)

a. Pose questions, such as

  1) How is the length of each bar used to represent a quantity?
  2) What is the meaning of each number on the vertical scale?
  3) Which bar shows the greatest number of pupils earning a particular salary?
  4) Which bar represents the mode? etc.
  5) How is a bar graph related to a scale drawing? (Note: the bars are simply lines made thicker.)

b. Guide pupils to an understanding of the following:

  1) A bar graph uses lengths of bars to represent quantities.
  2) The title of the graph tells us what it is about.
  3) The length or height of the bar is determined by the size of the quantity it represents.
  4) The scale of a bar graph always begins with zero.
  5) The choice of a scale depends upon the size of the numbers to be represented.
  6) The width of all the bars in a graph is the same.
  7) The distance between the bars is the same.

c. Suggested Practice

  1) Obtain appropriate material from newspapers, periodicals and texts.

     Note: An opaque projector would be a useful teaching aid.

  2) Select a bar graph from those collected by your class and tell all the information you can obtain from it.

2. Construction

a. Suggested Procedure

Draw a vertical bar graph to represent this information:

The number of pupils using our JHS library on each day of one week was as follows:

| Monday | Tuesday | Wednesday | Thursday | Friday |
|---|---|---|---|---|
| 120 | 180 | 200 | 160 | 140 |

Develop with pupils the following guides:

1) Plan the horizontal spacing of your graph

   a) Determine the number of bars, making the width of the bars equal and allowing equal spaces between them.

   b) Allow room at the left for the scale and its label.

2) Plan the vertical spacing

   a) Allow room at the top for the title
   b) Choose a scale such that the longest bar can be fitted on the graph, e.g. 20 (scale starts with zero)
   c) Allow room at the bottom for the names of the items to be represented according to the data.

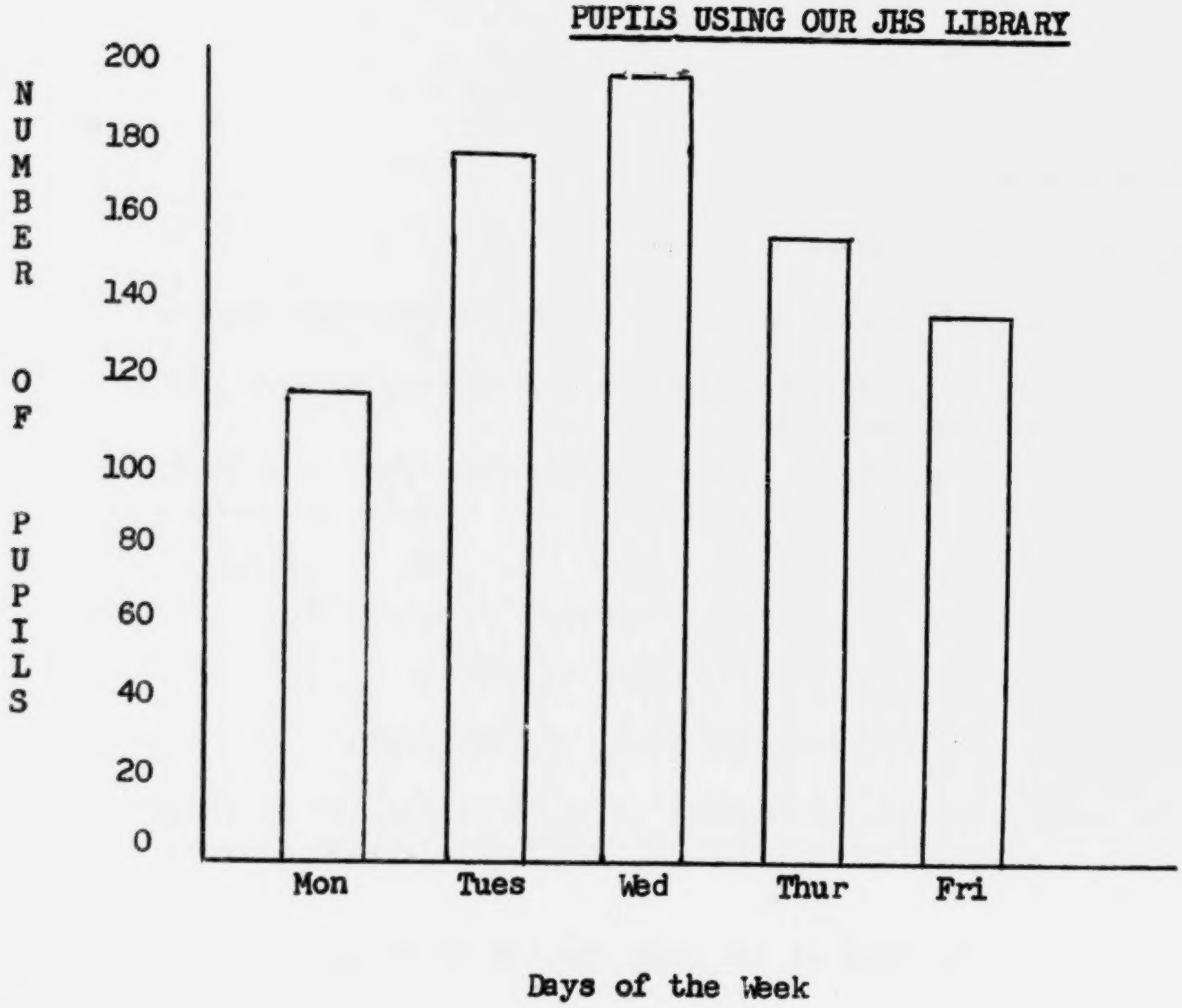
3) Decide on the order of items and write their names in the spaces at the bottom.

4) Mark off the scale on the lines at the left and label it.

5) Draw the bars according to scale.

6) Write the title of the graph at the top. This title should be appropriate and brief.

(See graph on following page)



b. Suggested Practice

1) If a bar in a graph extends from zero to halfway between 3,000 and 6,000 on the scale, what value does the bar represent?

2) If you are planning a bar graph on which the greatest value to be shown is 600 and you have ten spaces, what would your scale be? Why?

3) How can you decide what each space should represent in the scale of a bar graph you wish to make?

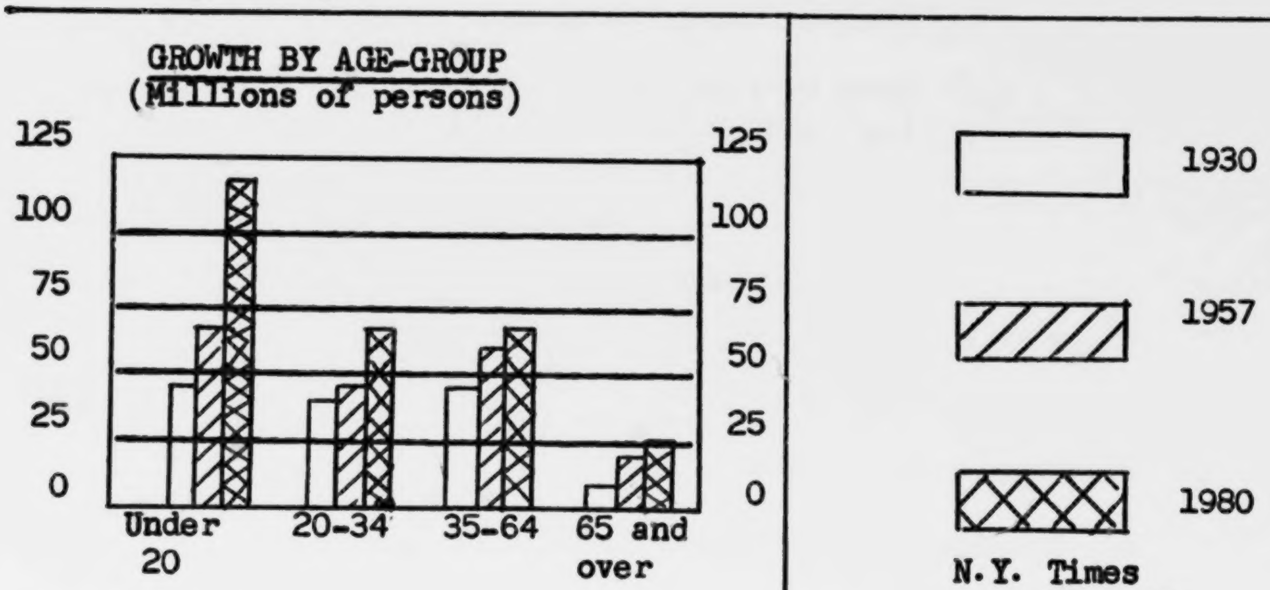
4) If you have decided that each space represents 50, where would the top of a bar representing 225 be?

5) Prepare a bar graph to represent the following information:

The airline distances from New York, N.Y. to some of the principal cities in the United States are as follows: Chicago, 713 miles; New Orleans, 1171 miles; Philadelphia, 83 miles; San Francisco, 2571 miles; Los Angeles, 2449 miles; Washington D.C., 205 miles.

Note to Teacher: Much of the illustrative material in this chapter and similar text materials lend themselves to use in the opaque projector or, by reproduction as transparencies for use in the overhead projector.

6) Present a graph such as the U.S. Population Growth by Age Groups.



N.Y. Times
Oct. 12, 1958
Pop. 175,000,000 -- Oct. 16,1958

7) Pose the following questions:

a) What was the population in each age group in 1930? 1957? What is the expected population in 1980?

b) What was the largest age group in 1930? In 1957? In 1980?

c) By how many millions of persons is it expected that the "under 20" age group will have increased between 1930 and 1980?

d) In which age group is the smallest increase expected between 1930 and 1980?

e) Of what value is this knowledge of expected population growth by age groups to a community which is planning a long range school building program?

8) Obtain appropriate material. Consult chapter on Statistics and Grade Texts.

9) Select a bar graph from those collected from newspapers and magazines and have pupils interpret the information you can obtain from it.

10) Show by a multiple bar graph the growth in height of boys and girls:

| | Boys Average Height | Girls Average Height |
|---|---|---|
| Age 6 years | 44" | 42" |
| Age 9 years | 53" | 51" |
| Age 13 years | 60" | 62" |
| Age 17 years | 69" | 63" |

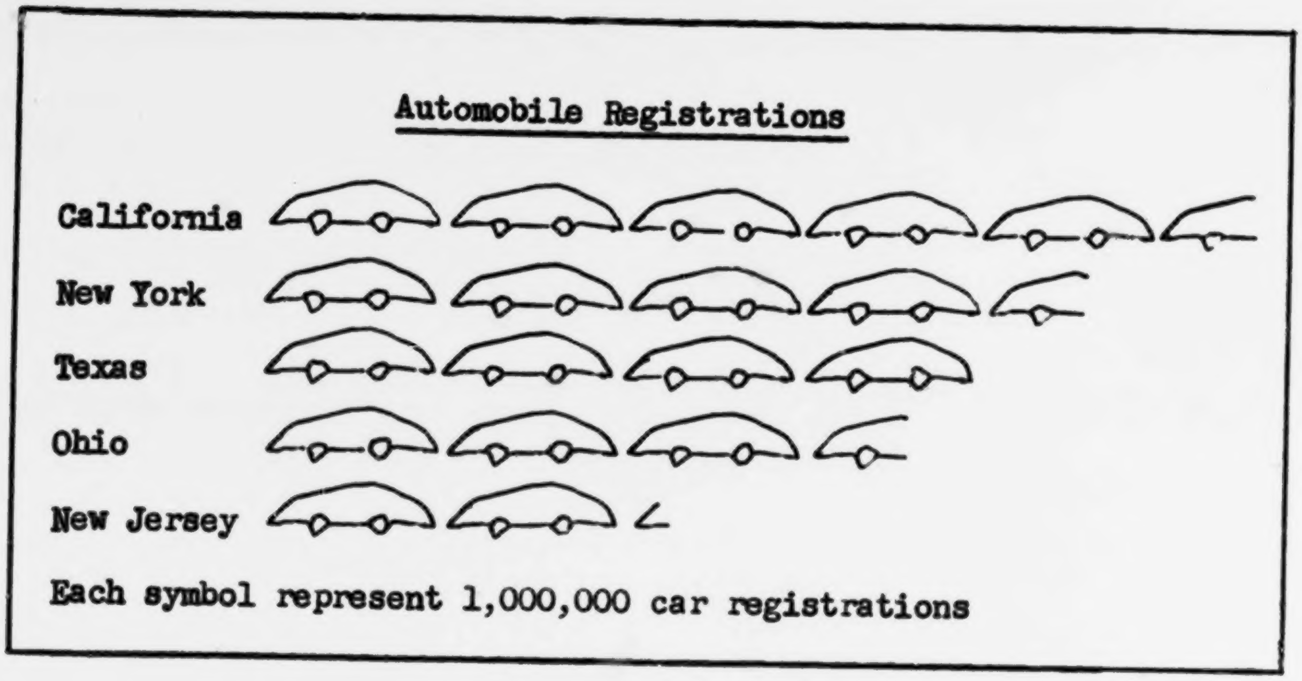
C. Pictograms (Picture graphs)

1. Reading and Interpretation

a. Suggested Procedure

1) Have pupils bring in examples of pictograms.

2) Present a pictograph such as the following for class discussion:



3) Pose the following questions:

   a) What is the meaning of each small car?

   b) State the information presented in the pictogram.

   c) Which state has the greatest number of registered automobiles?

   d) Which of the five given states has the smallest number of registered automobiles?

   e) About how many times the number of registrations in Ohio is the number of registrations in California?

   f) How can you find the approximate number of automobile registrations in each of the given states?

4) As pupils practice reading and interpreting pictograms, the following understandings should be developed:

   a) A small picture or symbol appropriate to the subject matter is used.

   b) Each symbol represents a given number of units. Different numbers are shown by more or fewer symbols.

   c) The lengths of the different series of symbols show comparisons just as the lengths of the bars do in a bar graph.

b. Suggested Practice

1) If one symbol represents 50 items, what number is represented by 7 symbols?

2) What is the ratio of two values if the first is represented by 12 symbols and the second by 4 symbols?

3) If each symbol stands for 3000 children, how many symbols would be needed to stand for 11, 816 children?

4) Each symbol in a certain pictogram represents 150 items. How many more items are represented by 6 symbols than by 4 symbols?

5) A table of information shows that there are 556 TV Stations in the United States. If a chart maker wishes to use no more than 11 symbols to represent this fact, what scale could he use?

6) Have children bring in pictograms from newspapers, periodicals, and the like for reading and interpretation.

Note: An opaque projector would be useful here.

2. Construction

a. Suggested Procedure

The following table shows the airline distances from New York, N.Y. Draw a pictogram to represent the facts.

|  | Airline Mileage from New York |
|---|---|
| Chicago | 713 |
| New Orleans | 1171 |
| Philadelphia | 83 |
| San Francisco | 2571 |
| Pittsburgh | 317 |
| Washington, D.C. | 205 |

1) Help pupil select a suitable symbol.

2) Guide him to examine the numbers involved to determine the numerical value which each symbol is to represent. For example, each symbol may represent 100 miles or 200 miles, or 300 miles. Discuss why 200 miles might be a good value for each symbol here.

3) Have pupil calculate the number of symbols needed for each item if one symbol stands for 200 miles. (Have pupil discuss how to represent the 713 mile distance to Chicago.)

4) Help pupil select an appropriate title (short and significant).

5) Have pupil draw the pictogram.

b. Suggested Practice

1) Show by a pictogram a comparison of the costs per mile for one person in the various modes of travel mentioned:

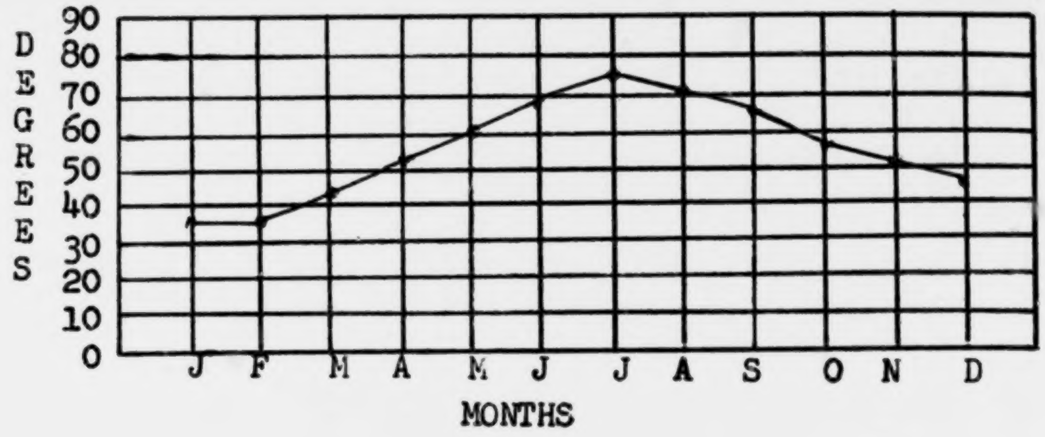
| | |
|---|---|
| Railroad Coach | 4¢ |
| Railroad First Class | $6\frac{1}{2}$¢ |
| Airplane | 7¢ |
| Automobile | 7¢ |

D. Broken-line Graphs

1. Reading and Interpreting

a. Have pupils bring in examples of broken-line graphs.

b. Present a broken-line graph such as the following for class discussion:



Average Monthly Temperature in New York City

(For 30-year period 1925-55)

c. Pose the following questions:

1) What is this line graph about?

2) What does the vertical scale represent? At what value does it begin? How are the numbers placed? What is the highest value on the scale? What is the value of each space?

3) What is shown on the horizontal axis?

4) What was the average temperature in April? in July? in December?

5) What was the lowest monthly temperature? In which month did this occur?

6) During which months was the average temperature more than 60°?

7) Why is this called a broken-line graph?

8) What does the graph tell us about the average monthly temperatures from January to July? from August to December?

d. As pupils practice reading and interpreting broken-line graphs, the following understandings should be developed:

1) The line graph shows different quantities by means of a dot located on a grid. The dot is associated with two values, one on the horizontal scale, and one on the vertical scale.

2) Changes in the direction of the line show quickly how the quantities vary and make it possible to see the trend of the information.

3) The vertical and horizontal axes are labeled.

4) The interval value or scale on each axis is shown.

5) Grid lines are numbered, not the spaces.

6) The numbers to be represented must be arranged in a systematic order, e.g., the months of the calendar year.

7) In a sense, the line graph is an extension of a bar graph in which the top points of the bars are joined by a straight line. The line graph is used when there are a good many values to be plotted and we may be interested in the trend.

e. Suggested Practice

1) On a line graph, what is the lowest value on the vertical axis?

2) How is the horizontal axis usually marked?

3) How is the vertical axis usually marked?

4) How is a point on a line graph identified and given a value?

5) Refer to a line graph brought in and answer questions similar to c above. In each case have pupil explain why the graph was more helpful than **figures showing the same** information.

2. Construction

a. Pose problem:

Picture by means of a broken-line graph the information given below concerning the speed of an automobile and the distance within which it can be brought to a stop.

| | | | | | |
|---|---|---|---|---|---|
| Speed in miles per hour | 20 | 30 | 40 | 50 | 60 |
| Average stopping distance | 42 | 80 | 125 | 185 | 250 |

b. Develop with pupils the following guides in drawing a line graph:

1) Draw a horizontal axis and a vertical axis which meet at right angles. The point at which they meet is called the origin.

2) Choose a suitable scale (25, 50, 75, etc.) for the information to be presented. Mark the vertical axis remembering to number the grid lines, not the spaces between these lines.

3) Prepare the horizontal axis with the other information (20, 30, 40, 50, 60) associated with the values to be represented.

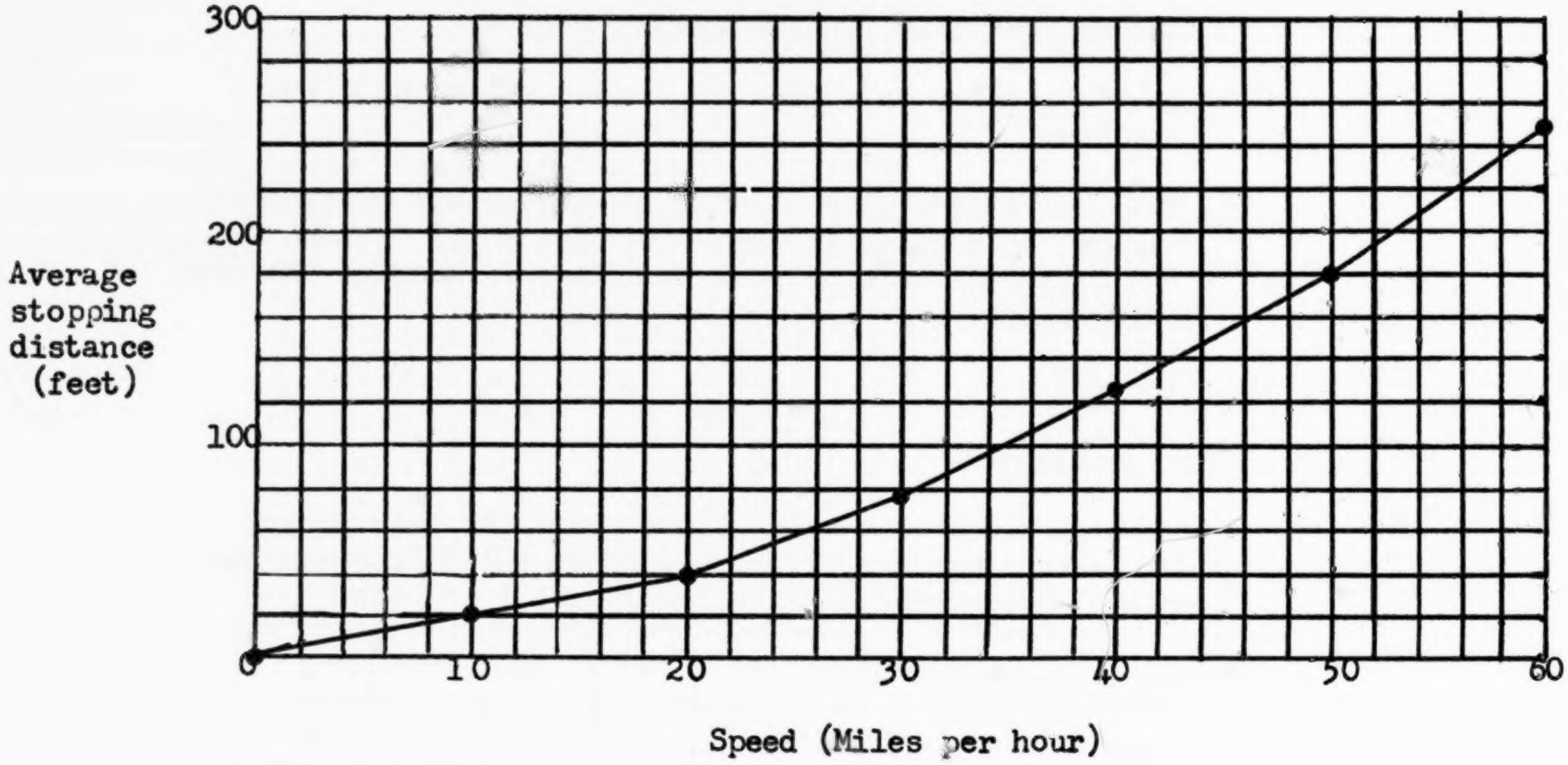
4) Show by a small dot or point, the value associated with each item of the information presented.

5) Begin at the left and draw a straight line from the first point to the second. Then draw a second line from the second point to the third, and so on until the last two points are connected by a straight line.

6) Label the axes and select a title for the graph.

Average Stopping Distance of Automobiles Moving

At Twenty to Sixty Miles Per Hour



c. Suggested Practice

1) The Science Department of Fairview Junior High School kept the following record of the temperature on a certain school day:

| Hour | Temperature | Hour | Temperature |
|---|---|---|---|
| 8 a.m. | 60° | 12 noon | 70° |
| 9 | 62° | 1 p.m. | 72° |
| 10 | 65° | 2 | 71° |
| 11 | 66° | 3 | 70° |

a) Make a graph showing these temperatures.

b) At what time was the temperature highest? At what time was the reading lowest? What were these temperatures?

c) During what hours was the temperature increasing? During what hours was it decreasing?

d) At what times was the temperature 70°?

e) What was the trend in temperature readings this day?

2) The following figures show the monthly rainfall or snowfall in inches at New York one year. Make a graph to show these facts:

| | | | |
|---|---|---|---|
| January | 2.3 | July | 3.5 |
| February | 4.7 | August | 2.9 |
| March | 5.2 | September | 1.8 |
| April | 3.2 | October | 4.3 |
| May | 2.2 | November | 3.0 |
| June | 3.5 | December | 4.2 |

3) Have pupil collect, record, organize and present by means of broken-line graphs such data as per cent of attendance for a period of time, number of books borrowed from the school library over a period of time, etc.

E. Rectangle Graph

1. Reading and Interpreting

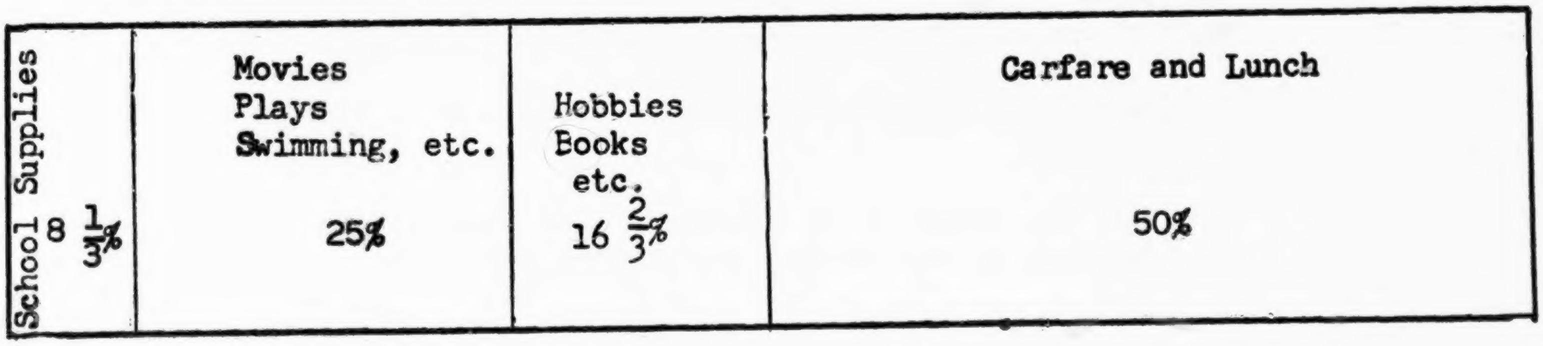
Note: Rectangle graphs are used to show the relationships of the various parts to the whole (the entire rectangle).

a. Have pupil bring in examples of rectangle graphs from newspapers and magazines.

Note: It is advisable to use an opaque projector so that all the pupils may view the graphs and join in the discussion.

b. Help pupil read and interpret these graphs, such as:

How Frank Spends His Allowance



1) What is the title of the graph?

2) What is the shape of this graph?

3) What per cent of Frank's allowance is represented by the whole rectangle?

4) What per cent of this allowance was spent on movies? on school supplies; on books? on carfare and lunch? on swimming, plays, etc.?

5) What part of the entire rectangle is the section marked carfare and lunch? movies, plays, etc.? books, hobbies, etc.?

6) How does this graph differ from a broken-line graph? from a bar graph?

7) When is this the best type of graph to use?

c. As pupils practice the reading and interpreting of the rectangle graph, the following understandings should be developed:

1) In a rectangle (single-bar graph), the entire length of the bar represents the whole amount, or 100%.

2) The size of each section depends on the quantity or part of the whole that it represents. Relative sizes rather than exact values are shown.

3) The sections may be readily compared with each other and with the whole.

2. Construction

a. Suggested Procedure

Pose problem:

Jack spends 6 hours in school, 3 hours at homework, 3 hours in sports, play, reading, 4 hours at meals, 8 hours sleeping. Make a rectangle graph of Jack's day.

1) Choose the length of the rectangle, e.g. 6 inches or 8 inches.

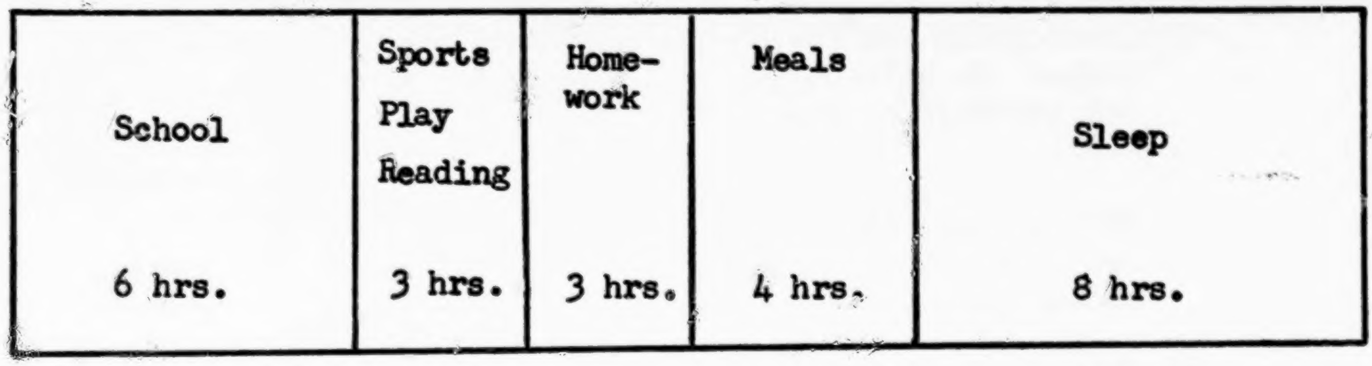
2) List the items to be included, the time for each, and the part of the whole each should take.

| Items | Time in Hours | Part of Day |
|---|---|---|
| School | 6 | $\frac{6}{24}$ |
| Sports, play, reading | 3 | $\frac{3}{24}$ |
| Homework | 3 | $\frac{3}{24}$ |
| Meals | 4 | $\frac{4}{24}$ |
| Sleep | 8 | $\frac{8}{24}$ |
| Total | 24 | $\frac{24}{24} = 1$ |

3) Draw the rectangle marking off the fraction parts:

Since 6" represents 24 hours, 1" represents 4 hours.

**Jack's Day**



4) Check that all items are represented and that the total of that which they represent adds up to the whole unit; e.g., the hours allotted should total 24 hours, a whole day; or in another example, the per cents should total 100%.

5) Place a short but appropriate title above the graph.

b. Suggested Practice:

1) Sam, a high school senior, works in a local drug store after school. He earns $15 a week. He saves 40% of his money, spends 20% on lunches, 30% on recreation and 10% on miscellaneous. Using a rectangle graph, show these facts.

2) There are 150 pupils in grade 9: 20 are in the newspaper club; 20 are in the art club; 50 are in the sports club; 40 are in the music club; 10 are in the mathematics club and 10 are in the photography club. Show this 9th year student club apportionment.

3) Similar problems may be taken from class texts, newspapers, and periodicals.

4) Show by a rectangle graph the sources of Estimated Federal Income: Individual Income Tax 58%; Corporation Income Tax 26%; Excise, estate, and gift taxes 15%; Customs receipts 1%.

F. Circle Graphs

1. Reading and Interpreting

a. Have pupils bring in examples of circle graphs.

b. Present a circle graph such as the one on the right for class discussion.

c. Pose the following questions:

1) Which grade has the largest enrollment? the smallest enrollment?

2) What is the total of the 3 per cents shown in the graph?

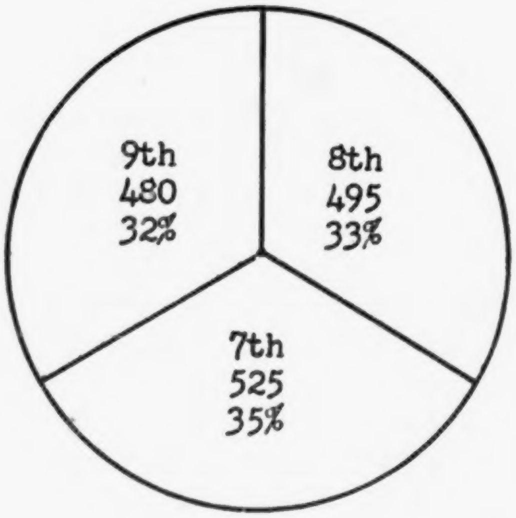
3) What is the total enrollment in the 3 grades?

4) What information does this graph provide?

5) What does the whole circle represent?

6) How could you estimate the relative sizes of the three grades without the printed per cents?

Avon Junior High School
Enrollment by Grades



d. **As pupils practice reading and interpreting circle graphs, the following understandings should be developed:**

   1) **The circle graph or pie chart is a circle with its inside divided into sectors. Each sector represents a part of the whole. The sectors may be compared with each other and with the total.**

   2) **When information is represented on a circle graph, relative sizes rather than exact values are pictured.**

   3) **The size of each wedge-shaped piece, or sector, is determined by the angle between the radii which form the central angle.**

e. Suggested Practice

   1) What fraction of a 24-hour day is spent at work by an office worker whose hours are 8:00 a.m. to 12 noon and 1:00 p.m. to 5:00 p.m.?

   2) If a circle graph were made of this worker's 24-hour day, what fraction of the circular area would be used to show his hours at work?

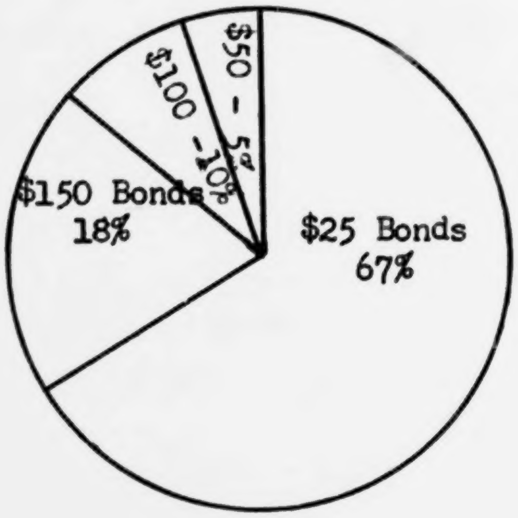
   3) How do you find the central angle you need to represent $\frac{1}{3}$ of a day? $\frac{1}{4}$ of a day? $\frac{1}{6}$ of a day?

   4) How many degrees are in the central angle needed to represent 2 hours of a 24-hour day? 8 hours? 12 hours?

   5) If the area bounded by a circle represents 1000 pupils, what fraction of this area should be used to represent 250 pupils? What per cent of the area bounded by the circle is this? How many degrees are in the central angle of the sector?

6) Use the graph at the right to answer these questions:

Sale of U.S. Savings Bonds by Denominations



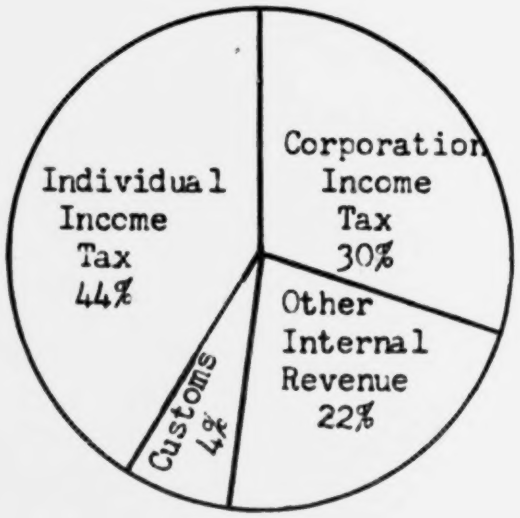
a) Which kind of bond was sold in the greatest number? Why?

b) About how many times as many $25 bonds as $50 bonds were sold?

c) How does the number of $100 bonds sold compare with the number of larger denomination bonds sold?

d) About what fraction of the total is the number of $50 bonds sold? of $100 bonds sold?

7) The graph at the right shows the tax revenue of Uncle Sam in one year. How much was taken in under each heading if the total income was 60 billion dollars?



2. Construction

a. Suggested Procedure

Pose problem:

Draw a circle graph to show the books that Bill read during the year for recreation. Bill reported that he read 12 fiction books, 12 dealing with science, 2 on travel, 10 in the biography and history area, and 4 others.

b. Develop with pupils the following guides in drawing a circle graph:

1) Total all the values to be graphed and find the part (or per cent) of the total represented by each value. Round large numbers to 2 or 3 places.

   12 + 12 + 2 + 10 + 4 = 40 books. (The whole is 40.)

2) Determine the number of degrees necessary to represent each value according to its part (or per cent) of the total.

3) Draw a circle and a radius. Begin with this radius and mark off in a clockwise direction the angle for the first item selected.

4) Continue to mark off the remaining angles.

5) Enter the title. Label each sector

Information obtained above may be presented in the form below.

| TYPE OF BOOK | NUMBER READ | FRACTIONAL PART OF 40 | FRACTIONAL PART OF CIRCLE | CENTRAL ANGLE |
|---|---|---|---|---|
| Fiction | 12 | $\frac{12}{40}$ or $\frac{3}{10}$ | $\frac{3}{10} \times \frac{360^\circ}{1}$ | 108° |
| Science | 12 | $\frac{12}{40}$ or $\frac{3}{10}$ | $\frac{3}{10} \times \frac{360^\circ}{1}$ | 108° |
| Travel | 2 | $\frac{2}{40}$ or $\frac{1}{20}$ | $\frac{1}{20} \times \frac{360^\circ}{1}$ | 18° |
| Biography | 10 | $\frac{10}{40}$ or $\frac{1}{4}$ | $\frac{1}{4} \times \frac{360^\circ}{1}$ | 90° |
| Others | 4 | $\frac{4}{40}$ or $\frac{1}{10}$ | $\frac{1}{10} \times \frac{360^\circ}{1}$ | 36° |
| Total | ? | ? |  | ? |

# VII

# INDIRECT MEASUREMENT

**The following materials and instruments of indirect measurement will be found useful in the implementation of the unit:**

**ruler, protractor, graph paper, steel tape, carpenter's rule (folding rule), alidade, hypsometer, simplified transit, Pythagorean Theorem device.**

## I. Concept of Indirect Measurement

### A. Distinguishing between direct and indirect measurement

1. Have pupil measure the length of his desk with a yardstick, the width of the room with a steel tape. Guide pupil to see that in each case we are applying the unit of measurement directly to the quantity to be measured.

2. Discuss with pupil whether a yardstick or a steel tape can be used to measure the distance from the earth to the moon. Have him see the impossibility of measuring this distance directly.

3. Have pupil suggest other situations in which direct measurement would be difficult or impossible. Discuss with pupil the necessity for making measurements (finding heights and distances) indirectly.

4. Discuss with pupils the importance of and the widespread use of indirect measurement in industry and science. Indirect measure is used to design a new airplane or automobile; to measure the distance to a star; to foretell the path of a satellite; to determine the depth of the ocean at a particular location; to foretell the water-level of a new ship when it is loaded.

5. Suggested Practice

   Which kind of measurement, direct or indirect, is needed to find:

   a. The height of a door in your home

   b. The distance to the sun

   c. The length of a rectangular kitchen table

d. The height of a mountain

e. Elicit from pupils other examples of each.

B. The essentials of indirect measurement

1. Have pupils suggest the names of instruments he knows that are used for indirect measurement, such as:

Clock (to measure time)
Thermometer (to measure temperature)
Spring Scale (to measure weight)
Odometer (to measure distance)
Altimeter (to measure altitude)
Transit (to measure heights and distance)

Note: When a measure of a quantity such as time, temperature, weight, velocity is computed from numbers obtained by actually measuring some other quantity (movement of gears, expansion of metals), the first quantity is measured indirectly.

2. Have pupil consider how measurements can be made indirectly and much more easily through the use of relationships expressed in formulas.

a. Guide pupil to see that:

1) The area of a floor can be found directly by counting the number of squares (square units) on the floor.

2) The amount of carpeting needed to cover a rectangular floor can be determined indirectly first by measuring the length and the width directly, and then by using the formula: $A = lw$.

3) Have pupil solve problems such as: How many square yards of carpeting are needed to cover a floor whose length is 15 feet and whose width is 12 feet.

b. Have pupil see that:

1) The circumference of a circular tablecloth can be measured directly by using a tape measure (obviously, a difficult thing to do).

2) The number of yards of lace needed to edge a circular tablecloth can be determined indirectly first by measuring the diameter directly and then by using the formula: $C = \pi d$.

3) Have pupil solve problems such as: How many yards of lace are needed to edge a tablecloth whose diameter is 70"?

c. Have pupil see that:

1) The volume of a box in the shape of a rectangular solid can be measured directly by counting the number of cubes (cubic units) it takes to fill it.

2) The volume of a packing case may be determined indirectly by measuring its length, width, height directly and by using the formula:

$V = lwh$

3) Have pupil solve problems such as: Find the volume of a packing case of length 36", width 24", and height 15".

3. Using the foregoing illustrations, guide pupil to note the essential elements of indirect measurement:

a. direct measurement

b. a relationship (formula)

c. computation

4. Have pupil consider how he would solve the following problems in indirect measurement:

a. How far away is a flash of lightning which is seen 5 seconds before the thunder is heard? Help pupil analyze the problem as follows:

1) The direct measurement is the time (5 seconds).

2) Since sound travels at approximately 1100 feet per second in the air, the relationship between the distance sound travels and the time of travel can be expressed as:

$d = 1100 \times t$ ($d$ = distance in feet; $t$ = time in seconds)

3) The computation is:

$d = 1100 \times 5$ or 5500.

The lightning is 5500 feet away.

b. What is the weight of a steel bar in the shape of a rectangular solid which is 12 inches long, 5 inches wide and 6 inches high? 1 cubic inch of steel weighs .28 pounds.

1) The direct measurements are the dimensions (12", 5", 6").

2) The relationship between the weight of the bar and its volume may be expressed by the formula:

$w = .28v$

where $w$ = weight in pounds and $v$ = volume in cubic inches.

3) The computation is:

$v = l \times w \times h$

$v = 12 \times 5 \times 6$ or 360 cubic inches

Since:

$w = .28v$

Then:

$w = .28 \times 360$ or 100.8

The steel bar weighs 100.8 pounds.

c. How high is a building if a rubber ball which is dropped from the roof takes 3 seconds to hit the ground?

1) The direct measurement is the time (3 seconds).

2) The relationship between the distance the ball falls (the height of the building) and the time is expressed by the formula:

$s = 16t^2$

where $s$ = distance (in feet) and $t$ = time (in seconds)

3) The computation is:

$s = 16t^2$

$t = 3, t^2 = 9$

$s = 16 \times 9 = 144$

The building is 144 feet high.

C. Suggested Practice

Have pupil indicate the essential elements of indirect measurement as he solves each of the following problems in indirect measurement:

1. Find the volume of the water in a tank 5 feet long, 2 feet wide and 3 feet high. Assume the tank is full.

2. The propeller blade of a certain airplane is 7 feet long. How far does the tip of the blade move in one complete revolution?

3. During rainy weather a heavy canvas covering is used to protect the baseball diamond used by a major league team. How many square yards of material are needed? (A baseball diamond is a square which measures 90' on a side.)

4. A man sees the smoke from the whistle of a train 4 seconds before he hears the sound of the whistle. How far away is the train?

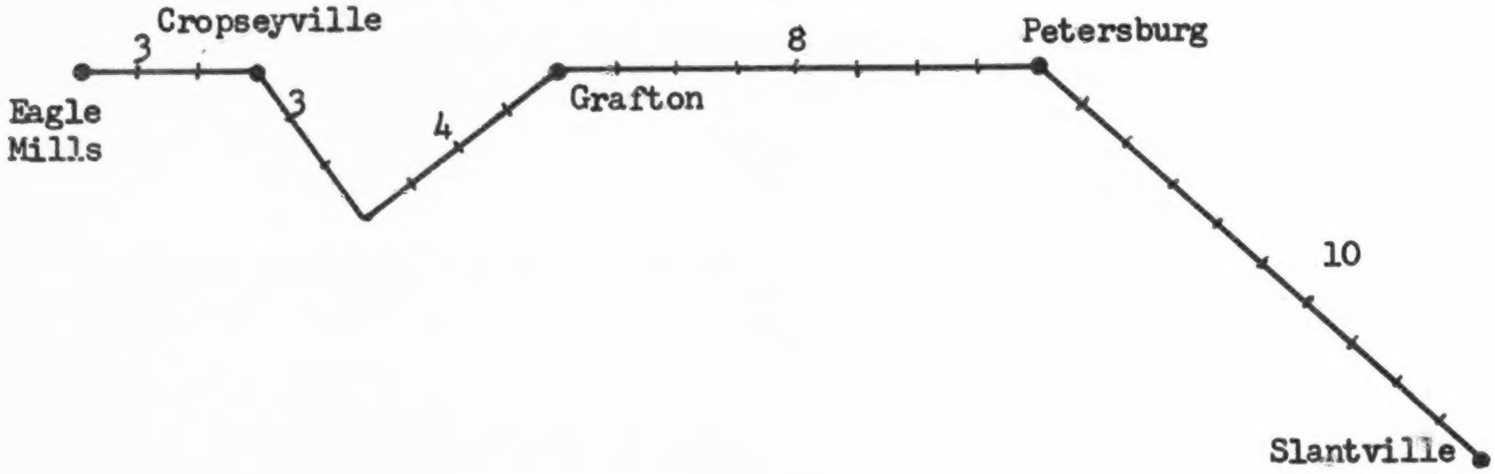
5. A skin diver descends to a depth of 20 feet below the surface of a fresh water lake. What is the pressure per square inch on his body? (Use the formula: $P = .43d$ where $P$ = pressure in pounds per square inch, and $d$ = depth in feet.)

6. Use the formula $s = 16t^2$ and find the height from which a freely-falling object fell if it took 2 seconds to reach the ground. ($s$ = the distance (height) in feet and $t$ = time in seconds.)

II. Scale Drawing

A. Concept of Scale Drawing

1. Have pupil examine the following drawing:



2. Have pupil count the number of units between Eagle Mills and Cropseyville. (3 units)

3. Tell pupil that the numeral 3 between Eagle Mills and Cropseyville means that there are 3 units between these two towns.

If each unit on the graph represents one mile on the ground, how many actual miles are there between Eagle Mills and Cropseyville as represented on the map? (3 miles)

4. Have pupil find the number of units between each of the following towns and tell how many miles are represented:

   a. Between Cropseyville and Grafton (7 units represents 7 miles)

   b. Between Petersburg and Slantville?

      ? units represents ? miles

5. Have pupil realize that the actual distances in miles are too large to place on paper. Therefore, the distances on paper are only representative distances. Tell pupils that a drawing of this kind is called a scale drawing.

6. Have pupil state the following actual distances as shown by the numerals on the road map.

   a. Between Cropseyville and Grafton

   b. Between Grafton and Slantville

   c. Between Eagle Mills and Petersburg

   d. Between Cropseyville and Petersburg

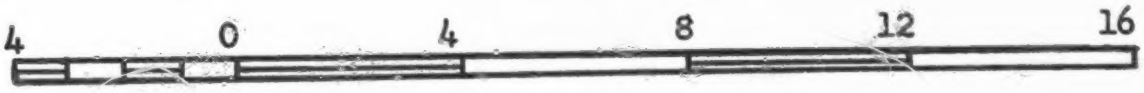
   e. Between Eagle Mills and Slantville

7. Tell pupils that the relationship that exists between a distance on a map and an actual distance on the ground is called a scale. The actual distance is drawn to scale.

B. Scale Drawings - Maps

Note: Prior to this section, have a pupil bring to class enough copies of one kind of road map for each pupil in the class.

1. Have pupil examine the drawing of the scale found on the map, e.g., the following $\frac{1}{4}$" unit:



1" = 4 miles

2. Have pupil observe the scale 1" = 4 miles.

   a. What does one inch on map represent on ground?

      (1" on map represents 4 actual miles on the ground, or 1" = 4 miles.)

   b. What would 2 inches on the scale represent on the ground?

   c. What do 4 inches on the map represent on the ground?

   d. What would $1\frac{1}{2}$ inches on the map represent on the ground?

   e. What distance would $2\frac{3}{4}$ inches on the map represent on the ground?

      If 2" = 8 miles and

      $\frac{3}{4}$" = 3 miles

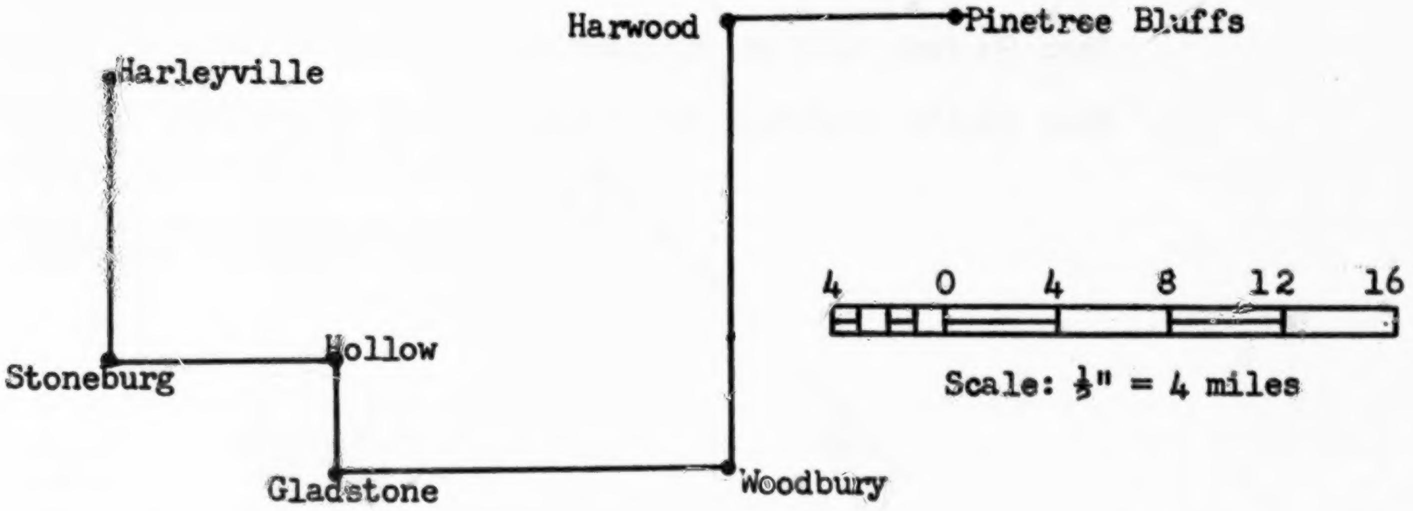
      Then $2\frac{3}{4}$" = 11 miles (distance on the ground).

3. Suggested Practice

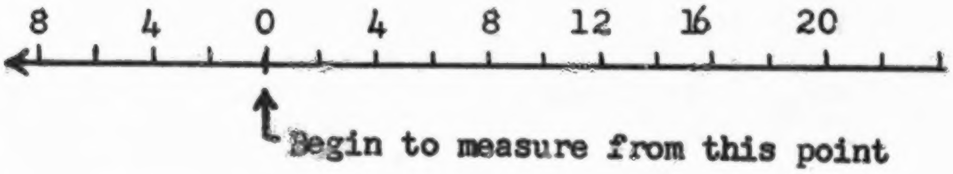
   Using above scale 1" = 4 miles, have pupil find the ground distances on a road map whose distances are:

| Distance on map | (Represents) | Distance on ground |
|---|---|---|
| $\frac{1}{2}$" | | ? |
| $1\frac{1}{2}$" | | ? |
| $3\frac{1}{4}$" | | ? |
| $2\frac{1}{2}$". | | ? |
| 4 " | | ? |

4. Have pupil draw a road map as follows:


Harwood
Pinetree Bluffs
Harleyville
4 0 4 8 12 16
Scale: ½" = 4 miles
Stoneburg
Hollow
Gladstone
Woodbury

a. Have pupil see that in lieu of a ruler to determine distances between points on the map, he may use a strip of paper on which the scale of the map has been duplicated.


8 4 0 4 8 12 16 20
Begin to measure from this point

b. Have pupil determine how many ground miles are represented by the distance from

1) Stoneburg to Hollow
2) Gladstone to Woodbury
3) Harleyville to Hollow
4) Woodbury to Harwood
5) Hollow to Woodbury
6) Stoneburg to Woodbury
7) Hollow to Pinetree Bluffs
8) Harleyville to Woodbury
9) Harwood to Gladstone
10) Pinetree Bluffs to Harleyville

c. Guide pupils to see that a road map drawn to scale is another example of the use of indirect measurement.

C. Scale Drawing - Layout of an Athletic Field

1. Constructing a Scale Drawing of a Baseball Diamond

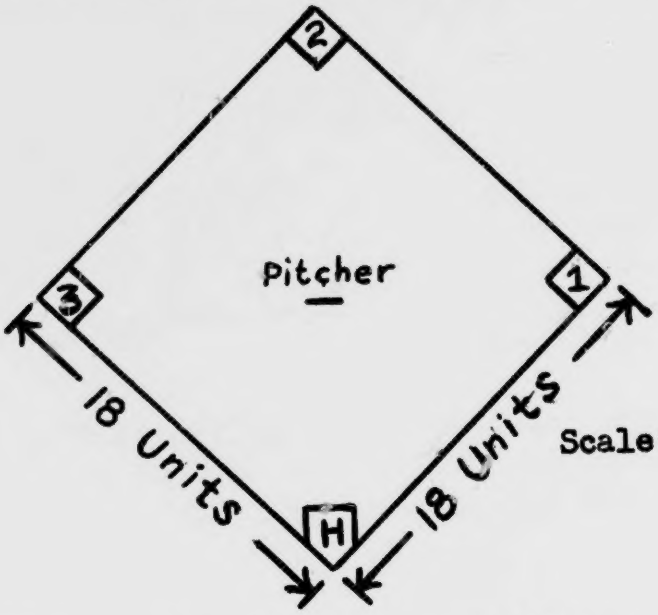
a. Elicit from pupil that a baseball diamond is in the shape of a square which measure 90 feet on a side. The pitcher's mound is about 60 feet from home plate.

b. Discuss with pupils an appropriate scale for use with $\frac{1}{10}$" graph paper, e.g.,

1 unit = 5 feet

Then 90 feet will be represented by $\frac{90}{5}$ or 18 units (1.8 inches).

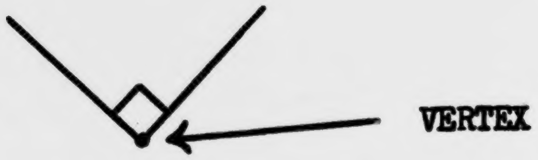
c. Have pupils construct the scale drawing as follows:



Scale: 1 unit = 5 ft.

1) Have pupils answer the following questions:

Note: Distances are measured from the vertex of the base.



Have pupils use a strip of $\frac{1}{10}$" graph paper to determine the answers to the following questions:

a) What is the distance from home plate to seecnd base? (Approximately 2.5" = 125')

b) What is the distance from first base to third base? (Approximately 2.5" = 125')

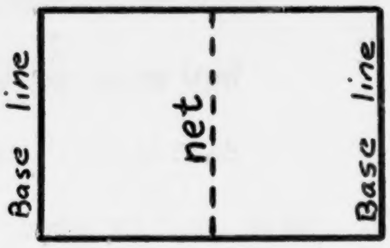
2. Suggested Practice

Have pupils construct a scale drawing of each of the following.

Guide pupils to use an appropriate scale.

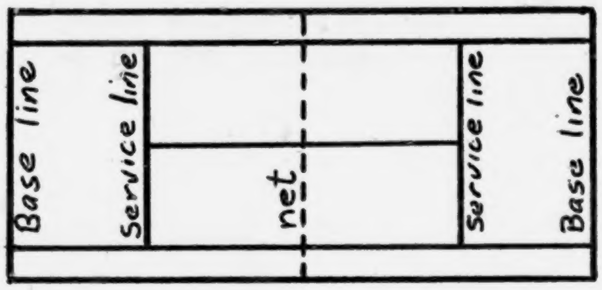
a. Volleyball court (rectangle in shape)

| | |
|---|---|
| Length of court: | 60 feet |
| Width: | 40 feet |
| Distance from base line to net: | 30 feet |


Base line
net
Base line

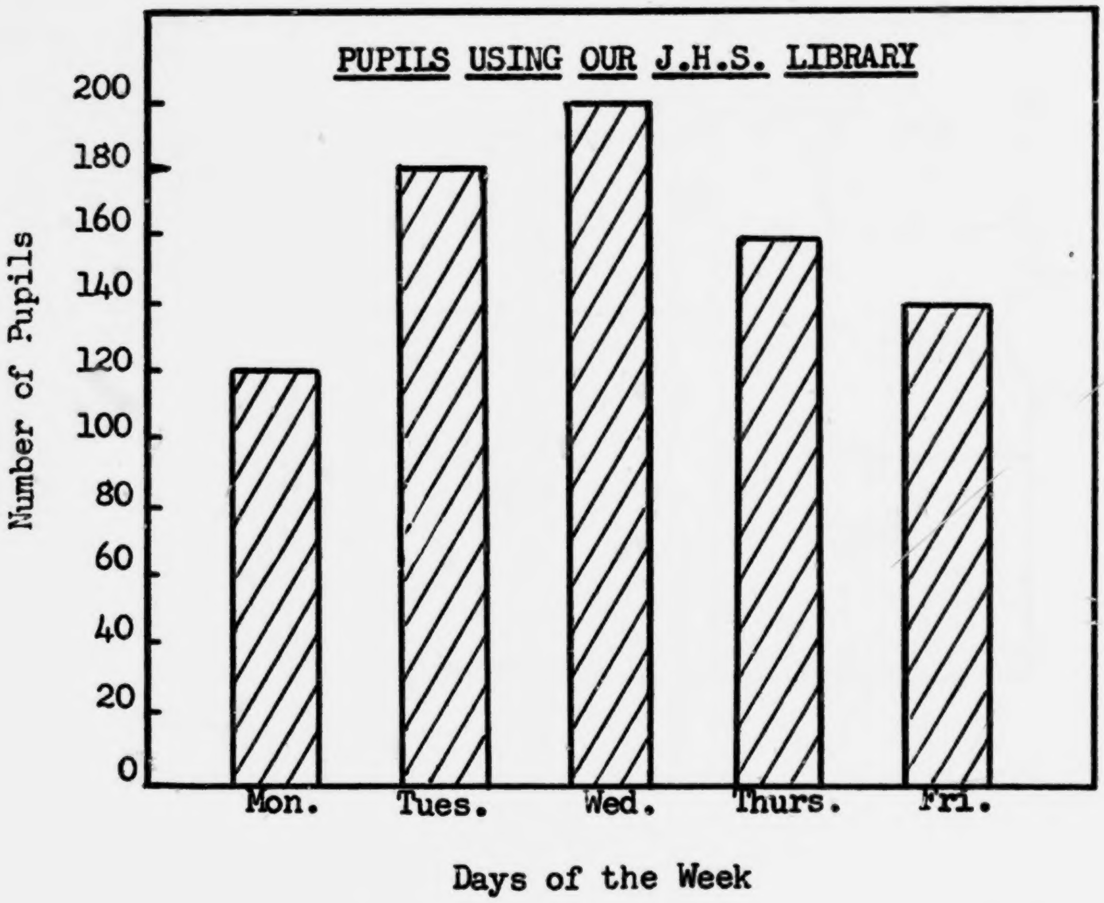
b. Tennis court (rectangle in shape)

| | |
|---|---|
| Length: | 78 feet |
| Width: | 36 feet |
| Distance from base line to service line: | 18 feet |
| Distance from base line to net: | 39 feet |


Base line
Service line
net
Service line
Base line

D. Scale Drawing: Graphs

1. Have pupils examine a bar graph such as the following:



a. Have pupil note that the length of each bar is drawn to scale. In the above graph (3 units = 20 pupils).

b. Pose following questions:

How many pupils use the library on Monday?

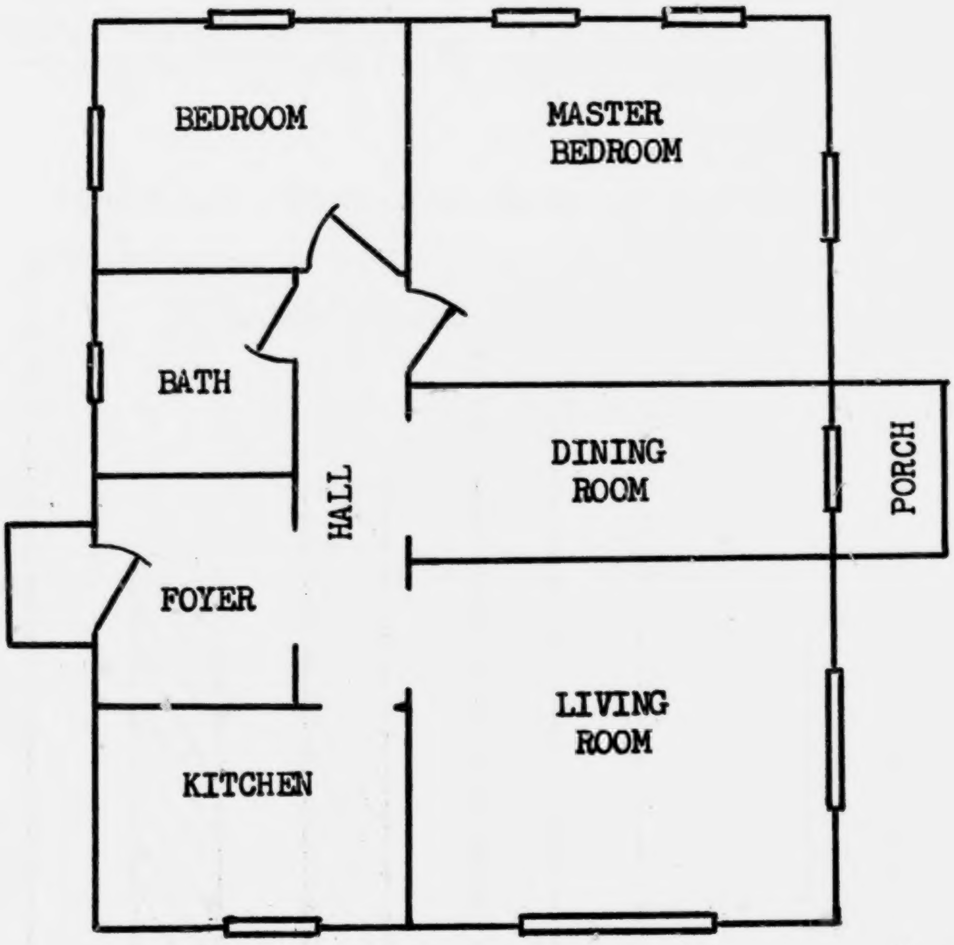
How many units represent that number of pupils?

Similarly, for the remaining days.

2. Have pupils study other types of graphs such as rectangle graphs, pictographs, line graphs to determine the scale of each graph.

E. Scale Drawing: Blueprints

1. Have pupils examine a floor plan such as the following or a similar one to be found in a textbook.


BEDROOM
MASTER BEDROOM
BATH
DINING ROOM
PORCH
HALL
FOYER
LIVING ROOM
KITCHEN

Scale: 1" = 8 feet

2. Have pupils answer the following questions:

   a. What are the dimensions of the living room?

   b. What are the dimensions of the master bedroom?

   c. How long is the hallway?

   d. What are the dimensions of the kitchen, etc.?

3. Have pupils determine the actual measurements of a room in his house and have him make a floor plan of this, using an appropriate scale.

### F. Scale Drawing: Dictionary

1. Have pupil look up general illustration of animals and insects in the dictionary and list each:

   a. Spider - full size

   b. Ant $1\frac{1}{2}$

   c. Millipede (thousand legger) $1\frac{1}{2}$

   d. Tse tse fly $1\frac{1}{2}$

   e. Elephant $\frac{1}{100}$

   f. Zebra $\frac{1}{50}$

   g. Giraffe $\frac{1}{100}$

   h. Flounder $\frac{1}{8}$

2. Have pupils notice the number that appears beneath the illustration. This number indicates the ratio of the height of the picture to the actual height. For example, the elephant as pictured in a dictionary has the number $\frac{1}{100}$ appearing beneath it. This means the pictured elephant is $\frac{1}{100}$ of the height of the actual elephant.

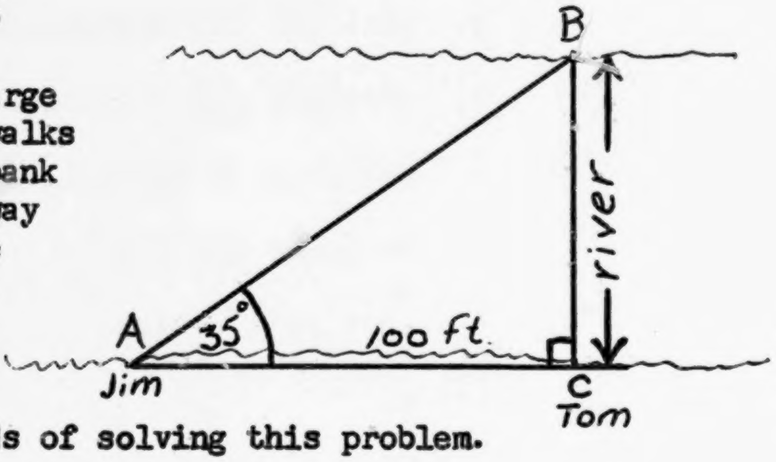
## III. Indirect Measure by Similar Triangles

Note: Similar triangles are introduced through scale drawings with which the pupil is now familiar. Through the development of an understanding of the proportion that exists between the corresponding sides of similar triangles, the pupils will be led to a new tool to use for indirect measure.

### A. Scale Drawing (see next page for scale drawing)

1. Pose problem:

   Tom and Jim stand on one side of a river and would like to know the width of the river without going across to measure. Tom stands on a bank opposite a large tree on the other side. Jim walks in a straight line along the bank away from Tom. At 100 feet away from Tom he measures the angle between Tom and the tree, and finds it to be 35°. Find the distance across the river.



   a. Have pupils suggest methods of solving this problem.

   b. Lead them to suggest a scale drawing as a means of indirectly measuring the width of the river.

2. Supply each student with a protractor and $\frac{1}{4}''$ graph paper for use in making a scale drawing. These drawings will be used in several lessons, therefore the pupils should keep them in their notebooks.

3. Divide the class into two sections A and B.

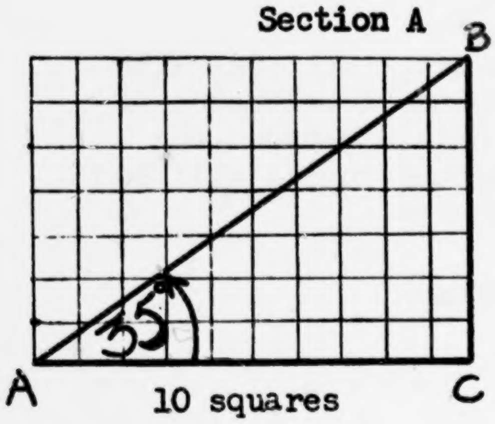
   Section A will use a scale - 1 small square - 10 feet ($\frac{1}{4}''$ = 10 feet)

   Section B will use a scale - 1 small square - 5 feet ($\frac{1}{4}''$ = 5 feet)

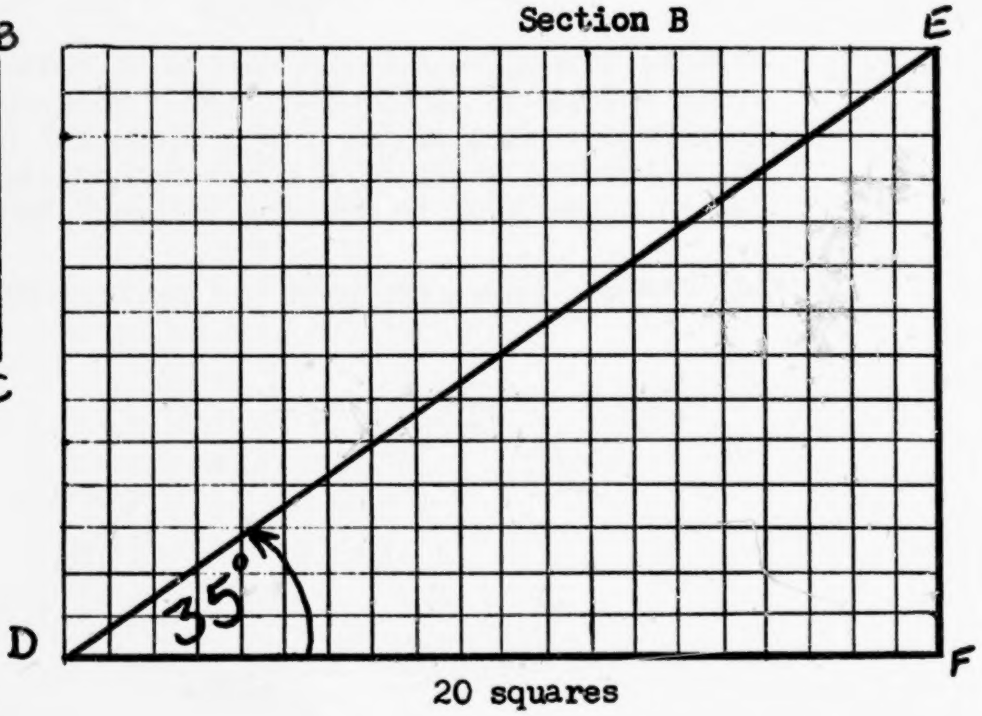
4. Using their assigned scales, have the pupils make a scale drawing of the problem. Warn the pupils to take great care in drawing an ∠ of 35°.

5. Section A will call their triangle ABC.
   Section B will call their triangle DEF.

Section A



Section B



6. Have the pupils read the width of the river from their drawings.

   Section A = 7 squares or 7 x 10 = 70 feet width of river.

   Section B = 14 squares or 14 x 5 = 70 feet width of river.

7. Elicit from the pupils that a change in the scale did not affect their answer for the width of the river.

   After a discussion as to why this is so, tell the pupils that these triangles represent a very special class of triangles called similar triangles.

B. Similar Triangles

When are two triangles similar?

1. Display a triangle from Section A and one from Section B.
2. Elicit from the pupils that
   a. these triangles have the same shape, although they are not the same size
   b. $\angle A = 35^\circ$ and $\angle D = 35^\circ$

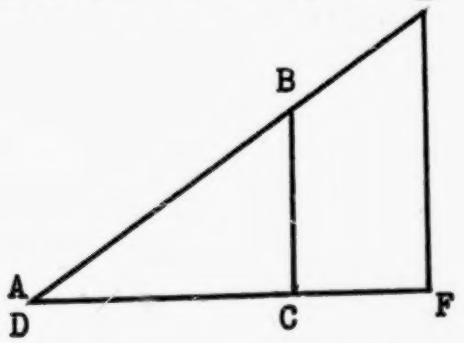
      $\angle C = 90^\circ$ and $\angle F = 90^\circ$

      $\angle B = 55^\circ$ and $\angle E = 55^\circ$
3. Tell the pupils that two triangles are called "similar" if each of the angles of one triangle (such as $\triangle ABC$) is equal respectively to the corresponding angle in the other triangle (such as $\triangle DEF$).

| $\triangle ABC$ | and | $\triangle DEF$ |
|---|---|---|
| $\angle A$ | = | $\angle D$ |
| $\angle C$ | = | $\angle F$ |
| $\angle B$ | = | $\angle E$ |

4. Tell the pupils that similar triangles always have the same shape.

   Cut out the two display triangles and by superimposing show that the triangles have the same shape.



5. The teacher might prepare several other right triangles such as $90^\circ$, $45^\circ$, $45^\circ$, and a $90^\circ$, $10^\circ$, $80^\circ$ and show the pupils that these do not have the same shape as their "scale" triangle of the river problem. Also that each of the angles of these triangles cannot be paired with a particular angle of their "scale" triangle.

   Pupils should conclude, therefore, that these triangles are not similar to their "scale" triangles.

C. Suggested Practice

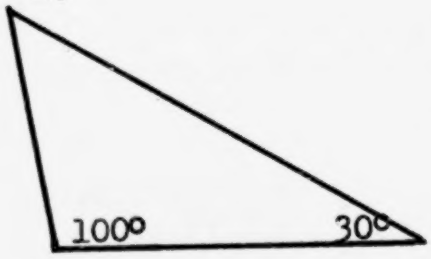
1. Review: The sum of the measures of the angles of a triangle equals $180^\circ$.

   Base angles of an isosceles triangle are equal.

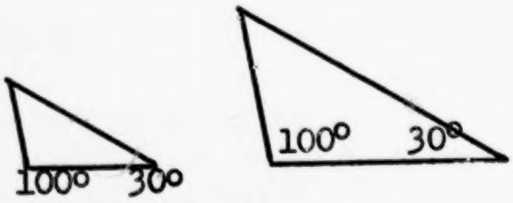
   All angles of an equilateral triangle are equal.

2. Tell which of these pairs of triangles are similar and why.

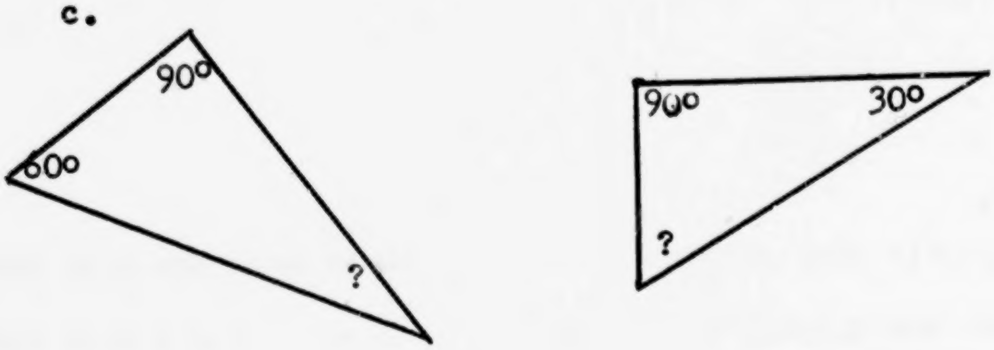
a.



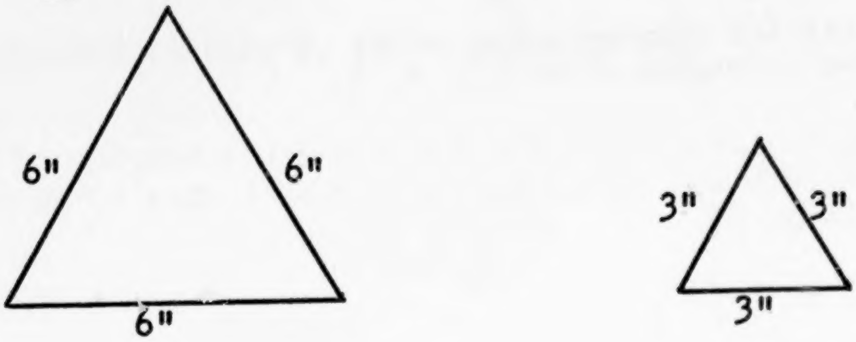
b.



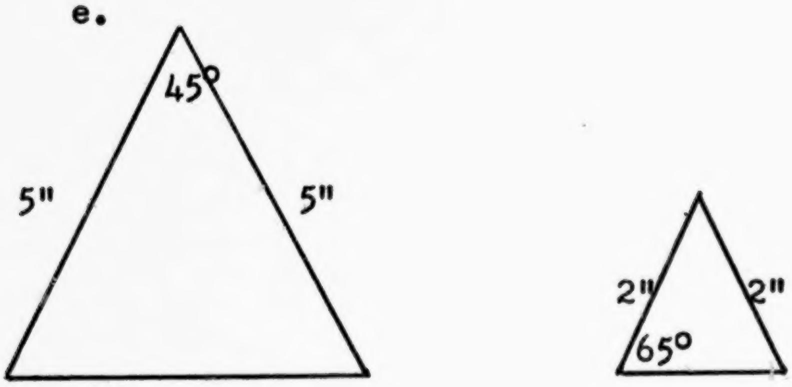
c.



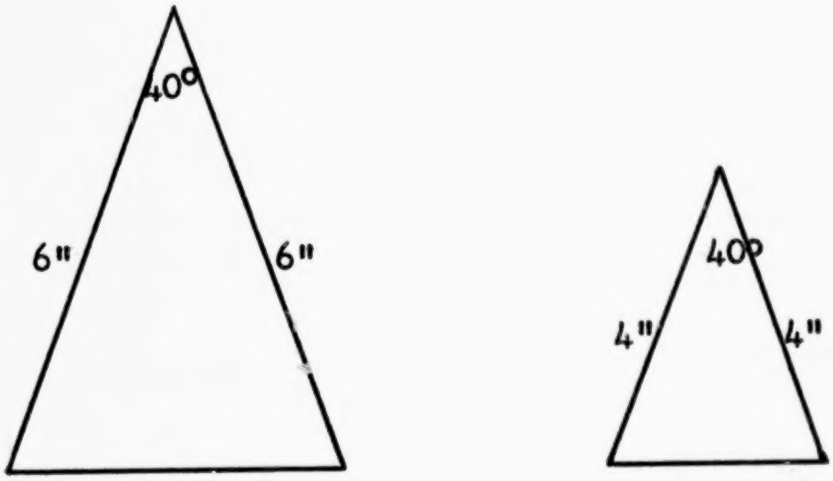
d.



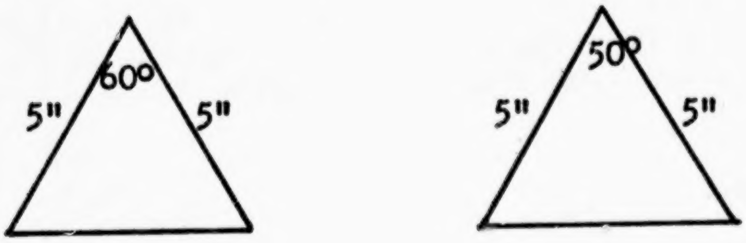
e.



f.



g.



D. Have pupil discover that the corresponding sides of similar triangles are in proportion. Use triangles from III, A, 5.

1. Have pupils compare the lengths of the bases and the lengths of the heights of triangles ABC and DEF. Tell pupils that side AC is called a corresponding side of DF.

   In $\triangle$ ABC; AC = 10 units

   In $\triangle$ DEF; DF = 20 units

   Then $\frac{AC}{DF} = \frac{10}{20} = \frac{1}{2}$

   In $\triangle$ ABC; BC = 7 units

   In $\triangle$ DEF; EF = 12 units

   Then $\frac{BC}{EF} = \frac{7}{14} = \frac{1}{2}$

2. Remind the pupils that when two ratios are equal to the same number as in (1) above, they can be set equal to each other. We call this a proportion.

   $$\frac{10}{20} = \frac{7}{14}$$

   Elicit from pupils the significance of the above, $\frac{10}{20} = \frac{7}{14} = \frac{1}{2}$

   (A side of triangle ABC = $\frac{1}{2}$ the size of the corresponding side of triangle DEF, and conversely, a side of triangle DEF = twice the size of the corresponding side of triangle ABC.)

3. Tell the pupils that when two triangles are similar, the corresponding sides are always in proportion.

4. Ask the pupils if they could represent the width of the river if the base were represented by a line segment 100 units long.

   Arithmetic Reasoning:

   In this case, the base will be 10 times the length of AC in $\triangle$ ABC. Therefore, the length of the line segment representing the width of the river will be 10 times the length of line segment BC in $\triangle$ ABC, or (10 x 7) = 70.

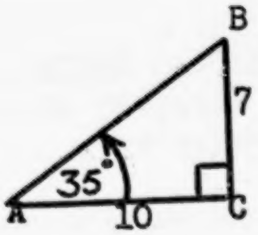
   Note: See diagram page 198.

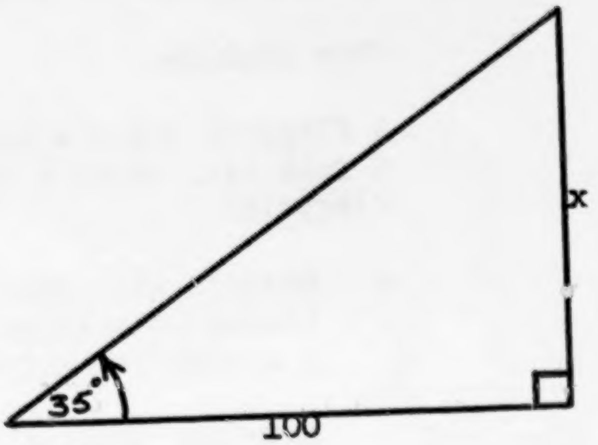
By algebra: Compare to △ ABC

$$\frac{x}{7} = \frac{100}{10}$$

$$x = \frac{7(100)}{10}$$

$$x = 70.$$



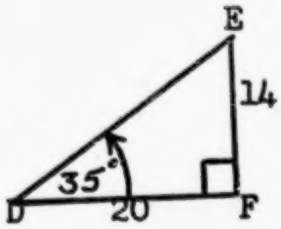


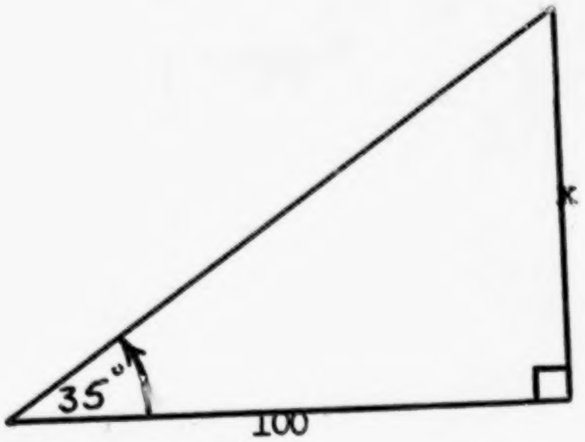
Compare to △ DEF

$$\frac{x}{14} = \frac{100}{20}$$

$$x = \frac{14(100)}{20}$$

$$x = 70$$





After several such problems, the pupils should realize that because the corresponding sides of similar triangles are in proportion, it is not necessary to make a scale drawing to solve such problems.

E. Solving Problems by Similar Triangles -- Shadow Method

1. Short History:

A story is told of Thales, a Greek mathematician, who lived about 600 B.C. On a trip to Egypt, he asked the height of the pyramids. No one knew. He astounded them by computing the approximate height by using a method which involved the shadow of the pyramid.

Today many Boy Scouts use the same method to find the height of a tree or building. This method is called "shadow reckoning". It is based on similar triangles.

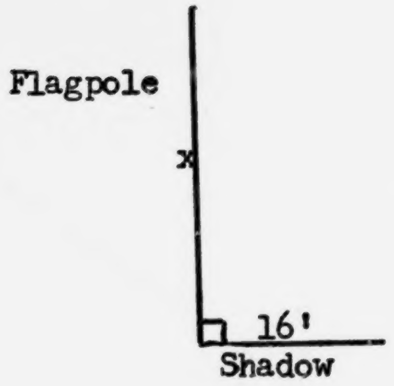
## 2. Shadow Method of Indirect Measure

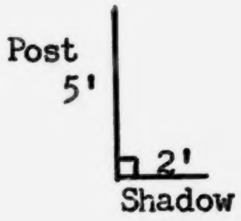
Pose problem:

A flagpole casts a shadow 16 feet long at the same time a post 5 feet tall casts a shadow 2 feet long. Find the height of the flagpole.

a. Discuss with the pupils the difficulty one would have in trying to measure the flagpole directly, and the need for a method of indirect measure.

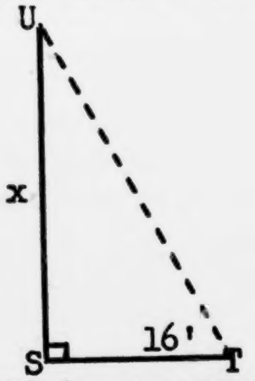
   Draw and label a diagram showing the conditions of the problem.

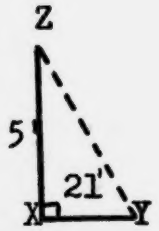




   1) Can we make similar triangles from these diagrams? (Draw in the required line.)

   2) Ask pupils to show that these triangles are similar by showing that the corresponding angles of the two triangles are equal.





   Note: At this point it will be necessary to show that the acute angles formed by the sun's rays on the ground at the end of the shadows are equal ( $\angle T = \angle Y$ ). Discuss with the pupils that because the sun is so far away, the rays are considered parallel near the earth. Therefore, the angles formed by these rays, as shown in the diagram, are equal.

   It should also be pointed out that as the sun goes down, the shadows get longer. It is for this reason that the problem must state that the 2 shadows were measured at the same time.

b. Have pupils set up the proportion between the corresponding sides of the two triangles.

$$\frac{\text{Height of Flagpole}}{\text{Height of Post}} = \frac{\text{Shadow of Flagpole}}{\text{Shadow of Post}}$$

c. Have pupils make substitutions from the diagram: $\frac{x}{5} = \frac{16}{2}$

d. Multiply both sides by 5: $5 \times \frac{x}{5} = \frac{16(5)}{2}$

$$x = \frac{80}{2}$$

$$x = 40$$

Answer: Flagpole is 40 ft. high

3. Suggested Practice

Note: Have pupils do the following problems - be sure they follow each of these steps:

a. Read problem through

b. Draw and label a diagram of the conditions of the problem.

c. Write out the ratio of corresponding sides as shown in diagram.

d. Substitute the values given in problem.

e. Solve the ratio equation.

1) How did Thales solve the problem of the pyramid?

Thales solved the problem of the pyramid in the following manner: He set up a stick of known length near the pyramid. He waited until the shadow of the stick equaled its length. He then measured the length of the shadow of the pyramid. This length he declared was equal to the height of the pyramid. Draw the diagram and explain the reasoning that Thales used to arrive at this conclusion.

2) An apartment building casts a shadow 75 feet long when a telephone pole 22 feet high casts a shadow 10 feet long. How high is the apartment building?

3) The shadow of a church steeple on level ground is 25 feet long. At the same time, an 8 foot lamp post casts a 5 foot shadow. What is the height of the steeple?

4) A tree casts a shadow 120 feet long. At the same time a man 6 feet tall casts a shadow of 8 feet. What is the height of the tree?

5) The sides of a triangle are 3, 10, and 8 inches respectively. The shortest side of a similar triangle is 12 inches. Find the lengths of the other sides of the second triangle.

4. What are some of the disadvantages to using the shadow method as a tool for indirect measure?

## IV. Trigonometry of the Right Triangle (optional)

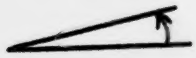
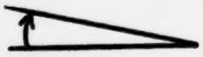
The use of the tangent ratio in indirect measurement.

### A. Suggested Procedure

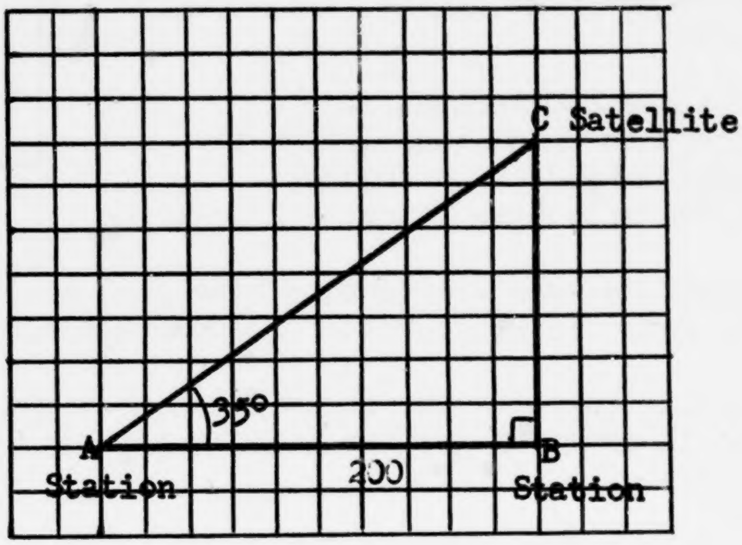
1. Pose problems such as this:

At the same time one tracking station observes a satellite directly overhead, another tracking station, 200 miles away, observes the angle of elevation of the satellite is 35°.

At that instant, what is the altitude of the satellite over the first station?

(Angles of elevation use the horizontal side as the initial side of the angle, i.e.  or  are angles of elevation).

a. Have pupils solve this problem by making a scale drawing such as the one that follows. Use $\frac{1}{4}$" graph paper.



Let A and B represent the positions of the tracking stations, and C the position of the satellite.

Have pupils draw the base line AB (10 units represent 200 miles)

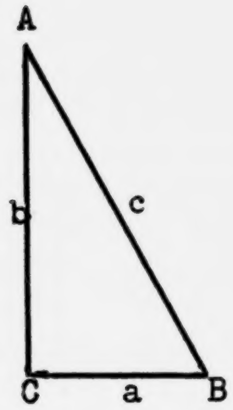
At A, using a protractor draw the angle of elevation (35°) and extend the terminal side to meet the perpendicular BC (which represents the altitude of the satellite) at C.

b. Discuss with pupils a serious disadvantage of scale drawings. A slight error in the drawing may result in a rather large error in the final result.

c. Tell pupils they are going to find a way of solving problems such as this without using scale drawings.

B. The Parts of a Right Triangle

1. Help pupils to identify the sides of a right triangle ABC in which C is the right angle.

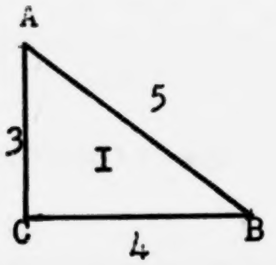


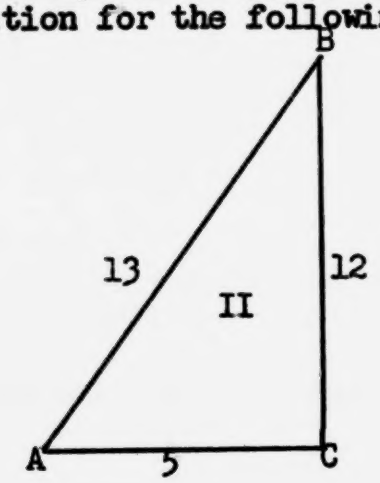
a. Have them identify the hypotenuse first, then the leg adjacent to (next to) one acute angle, then the leg opposite the same acute angle.

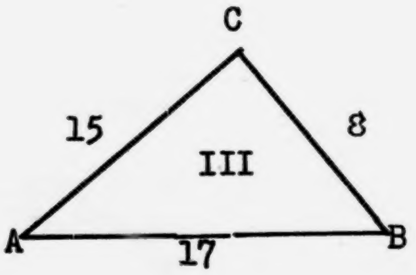
b. Side c is the hypotenuse, side a is adjacent to ∠ B, and side b is opposite ∠ B, side b is adjacent to ∠ A and side a is opposite ∠ A.

Note: Capital letters are used to name angles. Lowercase letters are used to name sides, and also to represent the lengths of the sides.

2. Have the pupils identify the sides of these right triangles and fill in the information for the following table, using the numerals on the diagram.







| | Hypotenuse | Side Opposite ∠ A | Side Adjacent ∠ A | Side Opposite ∠ B | Side Adjacent ∠ B |
|---|---|---|---|---|---|
| I | | | | | |
| II | | | | | |
| III | | | | | |

## C. The Ratio of Opposite Side to the Adjacent Side (The Tangent Ratio)

1. Have pupils refer to the problem in A-1 and use different scales in constructing right triangles.

   a. Let $\frac{1''}{4}$ represent 50 miles.

   b. Let $\frac{1''}{4}$ represent 40 miles.

   c. Let $\frac{1''}{4}$ represent 25 miles.

   d. Let $\frac{1''}{4}$ represent 10 miles.

2. Use the triangles above.

   Have pupils measure the number of units in the vertical lines to the nearest tenth, and complete the following table for the $35^\circ$ angle A

| Side Adjacent $\angle$ A | Side Opposite $\angle$ A | Ratio of: $\frac{\text{Opposite Side}}{\text{Adjacent Side}}$ |
|---|---|---|
| | | |
| | | |
| | | |
| | | |

   a. Have pupils note that in each case the heights were the same, even though the scales were different.

   b. Have the pupils find the quotient to two decimal places.

   c. Have pupils generalize that for a $35^\circ$ angle in a right triangle, the ratio of the opposite side to the adjacent side is approximately .70.

   d. Have pupils refer to the problem in A-1-a and show them the height could have been found by multiplying the adjacent side by .70.

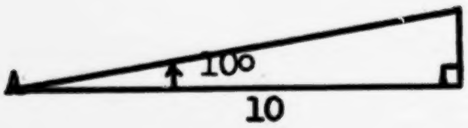
3. Have pupils construct a right triangle with legs each 10 units long; construct similar right triangles each with a pair of equal legs.

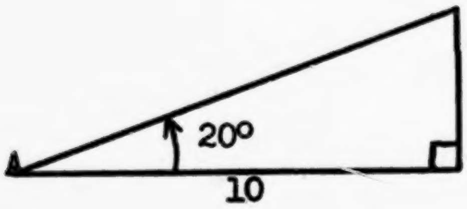
   Have pupil note that each acute ∠ is 45°. Have them arrive at the ratio of the opposite side to the adjacent side for a 45° angle in a right triangle is 1.00.

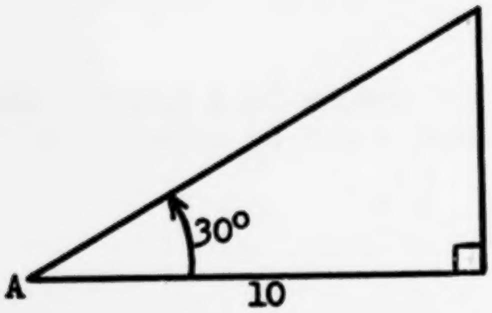
4. Have pupil learn that this ratio, of the opposite side to the adjacent side, which we call the tangent ratio, is constant for any one angle and the larger the ratio, the larger the angle.

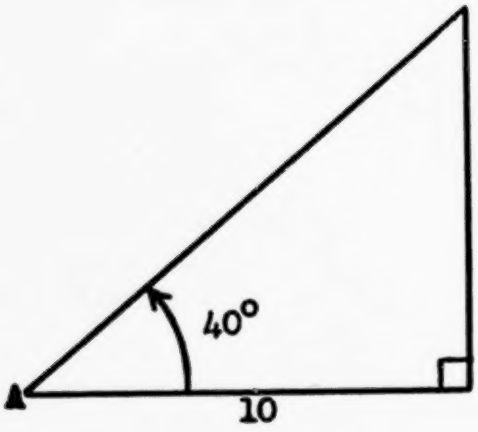
## D. Constructing a Table of Tangent Ratios for Some Angles Between 0° and 90°

1. Have pupils construct 8 different right triangles using acute angles of 10°, 20°, 30°, and so on; and one side side of 10 units. (See below)









2. Have pupils measure the vertical sides (the side opposite ∠ A) and complete the following table of values for the tangent ratio.

| Angle A | Side Opposite | Side Adjacent | $\frac{\text{Opposite Side}}{\text{Adjacent Side}}$ |
|---|---|---|---|
| 10° | 1.8 | 10 | .18 |
| 20° | 3.6 | 10 | .36 |
| 30° | | | |
| 40° | | | |
| 50° | | | |
| 60° | | | |
| 70° | | | |
| 80° | | | |

3. Have pupil generalize; the larger the angle, the larger the tangent ratio.

4. Have pupil note the tangent of 40° is not twice the tangent of 20°, and the tangent of 60° is not twice the tangent of 30°.

5. Have pupils compare the tangent table they have constructed in 2 above with the tables in their texts.

E. Using Tangent Tables in Indirect Measurement

1. Have pupils practice finding the tangent ratios of angles by using the table in a text.

2. Have pupils refer to the scale drawings they made in B-2, C-2 and have them realize that a small error in the use of the protractor is magnified in the reading of the vertical distance.

Guide the pupil to see that if we use the table of tangents it is not necessary to use a scale drawing. The tangent gives a much closer approximation of the true distance through computation.

3. Have pupils use the tangent to solve the following problem as shown:

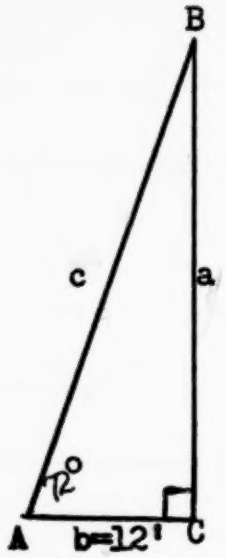
Pose problem:

A ladder, leaning against a building, forms an angle of 72° with the ground. If the foot of the ladder is 12 feet from the building, to the nearest foot, how high up on the building does the ladder reach?

a. Have pupils draw diagrams which approximate the conditions of the problem.

Note to Teacher: If the angles of the triangles are closely approximated, the relationships of the sides of the triangles will approximate the true conditions of the problem.

b. List the information given, and what is to be found.



b = 12'
A = 72°
Find length of side a.

c. Pupils should be led to realize that the side opposite and side adjacent are involved in the problem. Therefore, the tangent ratio may be used.

(1) $\text{Tan} \angle = \frac{\text{opposite side}}{\text{adjacent side}}$ Formula

(2) $\text{Tan A} = \frac{a}{b}$ Substitution

(3) $\text{Tan } 72^\circ = \frac{a}{12}$ Substitution

(4) $3.08 = \frac{a}{12}$ Substitution

(5) $(3.08)(12) = \frac{a}{12} \times 12$ Multiplication of both members by 12

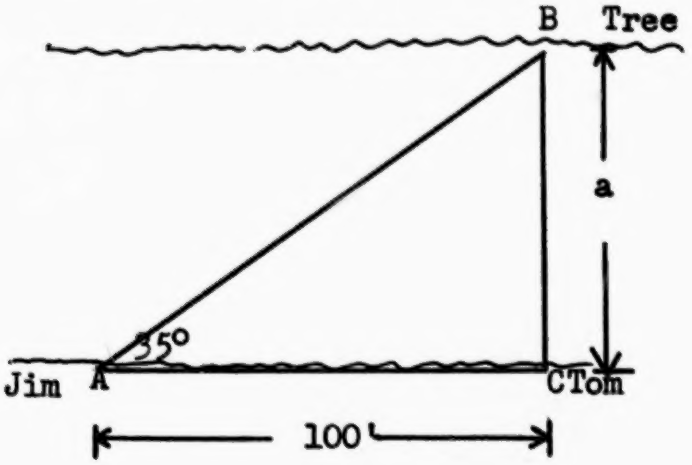
$36.96 = a$

(6) $37 = a$ Approximation

**Answer:** Length of side a = 37 feet.

F. Illustrative Problem:

1. Tom and Jim stand on one side of a river and would like to know the width of the river without going across to measure. Tom stands on a bank opposite a large tree on the other side. Jim walks in a straight line along the bank away from Tom. At 100 feet away from Tom he measures the angle between Tom and the tree, and finds it to be 35°. Find the distance across the river, to the nearest foot.



Given b = 100'
A = 35°

Find width of river

a. Have pupils draw the diagram making $\angle A$ approximately 35°.

b. Have them realize they know angle A, the side adjacent to A, and are trying to find the side opposite $\angle A$.

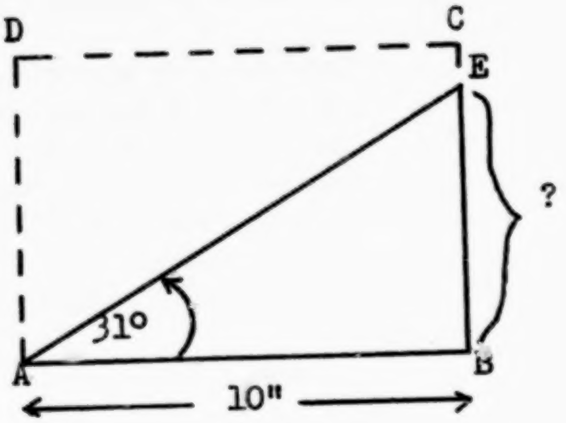
c. Have pupils write the formula:

| | |
|---|---|
| (1) $\text{Tan} \angle = \frac{\text{opposite side}}{\text{adjacent side}}$ | Formula |
| (2) $\text{Tan } A = \frac{a}{b}$ | Substitution |
| (3) $\text{Tan } 35° = \frac{a}{100}$ | Substitution |
| (4) $.70 = \frac{a}{100}$ | Substitution |
| (5) $100\,(.70) = \frac{a}{100} \times 100$ | Multiplication of both members by 100 |
| (6) $70 = a$ | |

Answer: Width of river = 70 feet

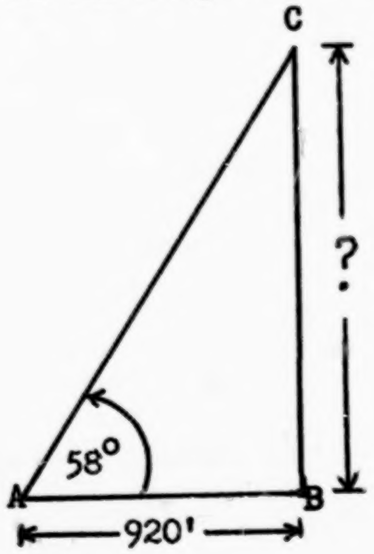
2. At a point 300 feet from the foot of a building the angle of elevation to the top of the building is 22°. Find the height of the building to the nearest foot. (Draw a diagram) (See IV, A, 1 for ∠ of elevation)

3. A mechanic has a rectangular piece of metal whose base measures 10 in. He has to cut it at an angle of 31° at the lower left corner. When cut, how long, to the nearest inch, is the right side of the triangular piece? (Find B E .)



4. A tree casts a 40 ft. shadow at a time when the sun's rays make an angle of 37° with the ground. What is the height of the tree to the nearest foot? Draw diagram.

5. To find the height (B C) of the Empire State Building, a point A was located 920 feet from the base B of the building. From A, the angle of elevation to the top of the building was 58°. Using this data, find the height of the building to the nearest foot.



## V. Indirect Measure by Means of the Pythagorean Rule

### A. Squares and square roots

1. Review the meaning of the exponent "2".

Pose Problem:

A square tile measures 9 inches on a side. What is its area?

a. Pupil recalls that to find the area of a square he multiplies the length by the width (here length equals width) so $9 \times 9 = 81$ square inches.

b. Have pupil find the areas of the squares whose sides are:

   $3$; $10$; $1$; $4.5$; $2\frac{1}{4}$.

c. Have pupil note that in each case he multiplied a number by itself. Have him review the short way mathematicians indicate that a number is to be multiplied by itself, e.g., $9 \times 9 = 9^2$.

d. Help pupil strengthen his understanding of exponent 2, i.e., $4^2$ tells him that 4 is to be multiplied by itself ($4 \times 4$).

e. Review with pupil that the area of a square may be expressed as $s^2$, where s represents the number of linear units in the side.

f. Practice:

   1. $4^2 = 4 \times ?$ $3^2 = ? \times ?$ $10^2 = ? \times 10$
      $r^2 = r\ ?\ r$ $c^2 = ? \times ?$

   2. Does $2 \times 5 = 5^2$? Explain.

   3. What numbers are to be multiplied in each of the following?

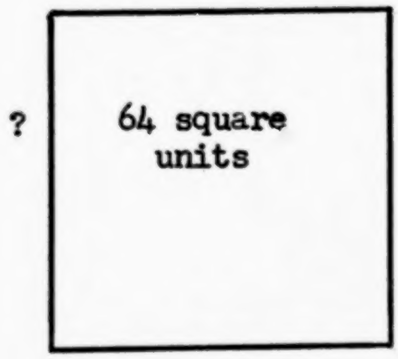
      $7^2$ $8^2$ $6^2$ $1^2$ $(3)^2$ $(\frac{1}{2})^2$ $(2.5)^2$

   4. Find the products in 3.

2. The meaning of square root

   a. Pose problem:

      What is the length of the side of a square whose area is 64 square units?



1) Have pupil recall that a number was multiplied by itself to get 64.

2) What number multiplied by itself gives 64? Pupil realizes that the length of the side of the square is 8 inches.

b. From diagrams of a number of squares, have pupil find the length of the side of each, given the area. (Use only squares whose sides are integers.)

c. Have pupil learn that 8 is one of the two equal factors of 64. One of the two equal factors of a number is called a square root of the number. The square root of 64 is written

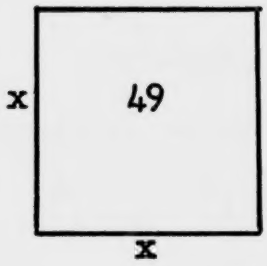
$\sqrt{64}$ . The symbol $\sqrt{\phantom{00}}$ is called a radical sign (from the Latin word "radix" meaning "root").

d. Practice:

1) What is the square root of 25? 49? 9?

2) Find: $\sqrt{16}$ $\sqrt{100}$ $\sqrt{36}$ $\sqrt{1}$

3) $x^2 = 49$ What is the value of x?



4) Enter the correct numerals in column marked x.

| $x^2$ | x |
|---|---|
| 1 | ? |
| 4 | ? |
| 9 | ? |
| 81 | ? |
| 144 | ? |

5) What is the area of a square whose side is 9? 81?

6) What is the side of a square whose area is 36? 100?

B. Have pupil learn to recognize a right triangle when the lengths of the sides are known.

1. Recognizing right triangles

a. Pose problem:

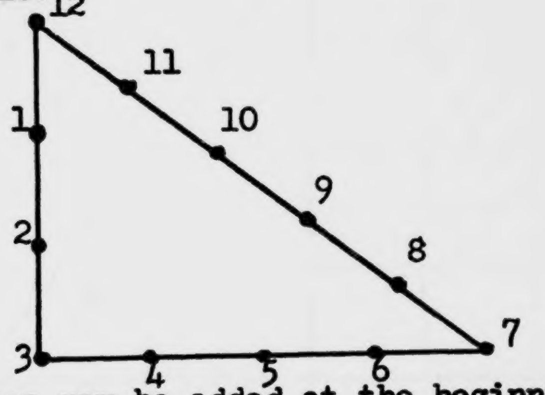
How did the Egyptians, who had few instruments, build their pyramids so that the base of each side is at right angles to the adjoining base? (Show picture of pyramid or model of square pyramid.)

1) Tell pupil about the Egyptian rope stretchers, who by tying ropes of twelve evenly spaced knots were enable to construct a right triangle.

2) Tie a rope as in the illustration below.

3) Have pupils form a triangle with sides 3 units long, 4 units long and 5 units long.

Have pupil see that a right triangle is formed. (Tie a knot at end of each unit, or color first three units one color, next four another color, and next 5 still another color.)



Note: Loops may be added at the beginning, the third, the seventh and the twelfth units so pupils can hold the rope more easily.

4) Have pupils conclude that a triangle whose sides are 3 units long, 4 units long and 5 units long is a right triangle.

Note: This may also be demonstrated with the peg board or the 6 ft. carpenter's rule.

b. Pose problem:

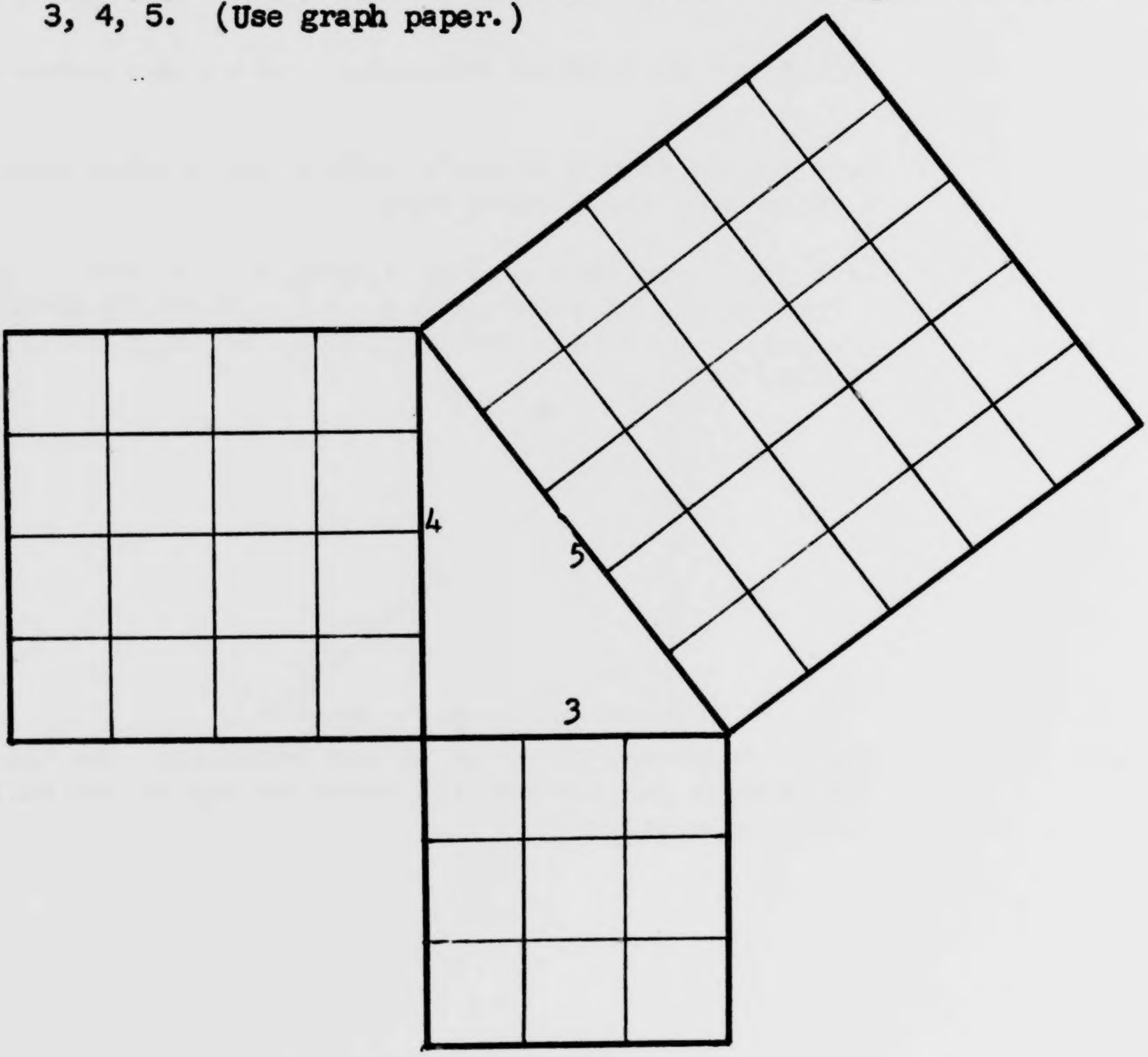
Is a triangle whose sides are 5 units, 12 units and 13 units a right triangle?

1) Pupils may suggest knotting a rope, or the peg board and string, or scale drawing. (demonstrate that it is a right triangle.)

2) Have pupil see that this is a lengthy process and that it would be desirable to find some general way of telling whether a triangle is a right triangle when the lengths of the sides are known.

C. Development of Pythagorean Rule

1. Have pupil build squares on each side of a triangle with sides 3, 4, 5. (Use graph paper.)



a. Have pupil find the area of each square by counting or by using the formula $A = S^2$ (9, 16, 25).

b. Help pupil find the relationship among the numbers 9, 16, 25. The sum of the squares on the legs of the right triangle, (9 and 16) is equal to the square on the hypotenuse (25).

c. Have pupil follow a similar procedure for a triangle whose sides are 5, 12, 13 units. The sum of 25 and 144 = 169.

d. Have pupil consider a triangle with sides 6, 8, 10. Is that a right triangle? Compare with the right triangle 3, 4, 5. He can see that the corresponding sides of this triangle are in the ratio 1:2 to the sides of 6, 8, 10 triangle. Therefore, 6, 8, 10 triangle is similar to the triangle 3, 4, 5 and so is a right triangle.

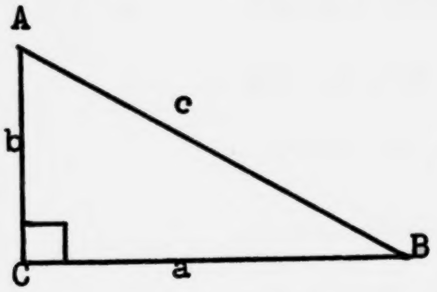
Have pupil test to see whether the sum of the squares on the arms (legs) of this triangle is equal to the square on the hypotenuse.

$6^2 + 8^2 = ?$

$36 + 64 = 100$

e. Have pupil conclude that in a right triangle the sum of the squares on the arms (legs) is equal to the square on the hypotenuse.

2. Help pupil express this relationship in a formula:



$a^2 + b^2 = c^2$

Note: In this formula the letter c represents length of the hypotenuse, the side opposite the right angle C.

3. Tell pupils that this formula is called the Pythagorean Rule and is named after the Greek mathematician Pythagoras, who discovered this relationship in the sixth century B.C.

4. Have pupils investigate to see whether the formula holds for triangles other than right triangles.

   a. Use peg board and string to attempt to lay out a right triangle with sides 2, 5, 6. (Use a strip of graph paper to represent hypotenuse -- 6 units.)

   b. Pupil sees this is not a right triangle.

   c. Have pupil square the numbers representing the sides (4, 25, 36), to see if the sum of any two squares equals the third.

   d. Have pupil try a few other problems of similar nature.

   e. Have pupil conclude that only in a right triangle does the sum of the squares on the legs equal the square on the hypotenuse.

   f. Optional - Have pupil conclude that if the sum of the squares on two sides of a triangle is not equal to the square on the third, the triangle is not a right triangle.

5. Practice:

   By means of the formula decide which of the following are right triangles:

   a. A triangle with sides 4, 5, and 6 units (use formula)

   b. Sides 6,6, and 6 units

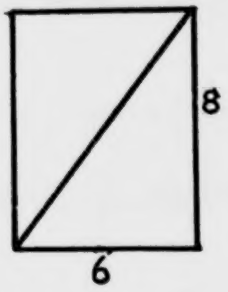
   c. Sides 9, 40 and 41 units

   d. Sides 8, 15, 17 units

   e. Sides 10, 24, 26 units

   f. Sides 12, 16, 20 units

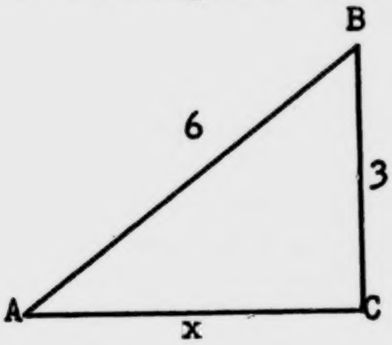
g. A door to a barn is 6 ft. by 8 ft. It is braced diagonally by a strip of wood. What is the length of the diagonal strip?



D. Finding the square roots of numbers by means of a table of square roots.

Pose problem:

A right triangle has a side of 3 inches and the hypotenuse is 6 inches long. What is the length of the third side?



1. Have pupil solve problem

| | |
|---|---|
| $a^2 + b^2 = c^2$ | Pythagorean Rule |
| $3^2 + x^2 = 6^2$ | Substitution |
| $9 + x^2 = 36$ | |
| $9 - 9 + x^2 = 36 - 9$ | Solution by subtraction |
| $x^2 = 27$ | |

Then x equals the square root of 27.

2. Tell pupil that mathematicians use this symbol $\sqrt{\phantom{27}}$, called a radical sign, to mean the square root. So the square root of 27 is written ($\sqrt{27}$).

3. The pupil will realize that he knows no number which when multiplied by itself will give a product of 27.

4. Have pupil learn to use the table of square roots in his textbook.

   a. Have him note that there are three columns listed:

| Number | Square | Square Root |
|---|---|---|
| 1 | 1 | 1.00 |
| 2 | 4 | 1.41 |
| 3 | 9 | 1.73 |
| 4 | 16 | 2.00 |

   Note: Have pupil use tables of roots to two places. If tables of three or four places are available, have more able pupils round the roots to hundredths place.

   b. Have him find the answer to the problem in C-1 by the use of the table.

   c. Have him check by multiplying the value given in the table (5.20) by itself. The product is close to 27.

   Note: Only an approximation of the square root of 27 is given in the table. Thus, the square root of 27 in the table is given as 5.20 and 5.20 x 5.20 is 27.04.

   d. Have pupil find the square roots of the following numbers: 12, 20, 36, 54, 125.

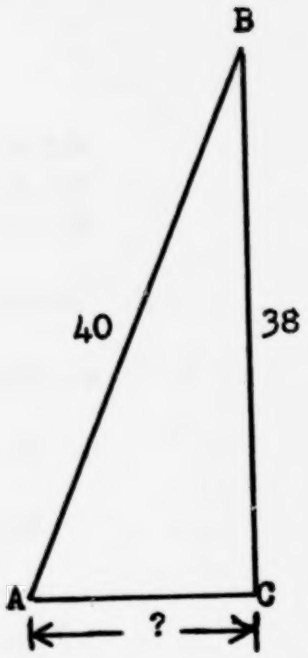
   e. Have him multiply the square root by itself in each case to discover how close the approximation is.

5. Practice: Use Tables of Square Roots

   Have pupil draw a diagram for each problem.

   a. A fireman's ladder is 40 ft. long. It is placed against the wall of a building 38 feet above the ground. How far from the building is the foot of the ladder? Illustrative solution.

   (Illustration on page 222)



$a^2 + b^2 = c^2$ Pythagorean Rule

$38^2 + x^2 = 40^2$ (Use table)

$1444 + x^2 = 1600$

$1444 + x^2 - 1444 = 1600 - 1444$

$x^2 = 156$

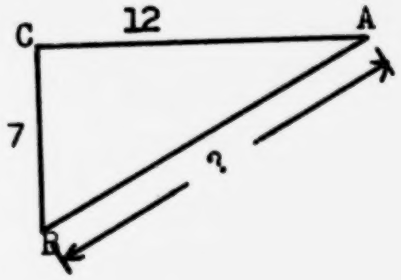
$\sqrt{x^2} = \sqrt{156}$

$x = 12.49$ (Use square root table)

The foot of the ladder is approximately 12 ft. from the building.

b. What is the length of the diagonal of a rectangle whose length is 13 feet and width is 5 feet.

c. A 17 foot ladder is placed against the wall of a building in such a way that it reaches a window 15 feet above the ground. How far is the foot of the ladder from the building?



d. In the figure to the right, A, B and C are the locations of three towns on a map. Town A is 12 miles due east of town C and town B is 7 miles due south of town C. Find the shortest distance between towns A and B.

e. Using the folding rule, have the pupil find the length of the hypotenuse if one side of a triangle is equal to 6 inches and the other to 12.

E. Methods of finding Square Roots

1. Using Square Root Tables

a. Using tables of square roots found in any text, have the pupils find the square roots of the following:

1) 1
2) 2
3) 3
4) 10
5) 16
6) 32
7) 40
8) 75
9) 100
10) 120

Which of the above numbers are perfect squares?
Which of the above numbers are not perfect squares?
Which of the above numbers are multiples of perfect squares?

2. Method of Approximation

a. Problem: Find the square root of 70, to the nearest tenth.

Is this number a perfect square:

Elicit from pupil: 8 x 8 = 64 (too little)
9 x 9 = 81 (too large)

How may we find the square root of 70 without using a table of square roots?

Lead pupils to conclude:

1) Square root of 70 lies between 8 and 9

2) 70 is closer in value to 64 than 81; therefore, try 8.3 as the square root.

3) Try

$$\begin{array}{r} 8.3 \\ \text{x}8.3 \\ \hline 249 \\ 664\phantom{0} \\ \hline 68.89 \end{array} \quad \text{(too little)}$$

4) Try

$$\begin{array}{r} 8.4 \\ \text{x}8.4 \\ \hline 336 \\ 672\phantom{0} \\ \hline 70.56 \end{array} \quad \text{(too large)}$$

5) However, 70.56 is closer to 70 than 68.89. Therefore, we will accept 8.4 as the closer approximation for the square root of 70.

b. Problem: Find the square root of 289.

Since the pupil may not recognize the square root immediately, elicit the estimate as follows:

10 x 10 = 100 (too little)
20 x 20 = 400 (too large)

Lead pupil to conclude:

The square root of 289 lies between 10 and 20. Since 289 is closer to 400 than 100, therefore the square root of 289 is closer to 20 than 10.

Since 289 ends in a "9" we need a number between 10 and 20 such that the product of the units digit multiplied by itself produces a 9. What are such numbers? (13, 17)

As estimated above, 17 is closer to 20 than 13. Try squaring 17.

$$\begin{array}{r} 17 \\ \times\ 17 \\ \hline 119 \\ 17\phantom{0} \\ \hline 289 \end{array}$$

; Therefore 17 is the square root of 289.

What is the relationship between 289 and 17?

3. Finding the square roots of numbers by means of the method of dividing, then averaging quotient and divisor (Newton's Method).

   a. Question: How would you find the square root of 289 if you had no table of square roots?

      1) Have pupil recall that he must find two equal numbers which when multiplied will give 289.

         Estimate: Between 10 and 20 ( 10 x 10 = 100
         20 x 20 = 400)

      2) Method

         a) Try 15

$$\begin{array}{r} 19 \\ 15\overline{)\,289} \\ 15\phantom{0} \\ \hline 139 \\ 135 \\ \hline \end{array}$$

15 and 19 are not equal numbers

         b) Take average of 15 and 19. $\frac{15 + 19}{2}$ (17)

$$\begin{array}{r} 17 \\ 17\overline{)\,289} \\ 17\phantom{0} \\ \hline 119 \\ 119 \\ \hline \end{array}$$

c.) Since the divisor and the quotient are each 17, the square root of 289 is 17.

d.) Check

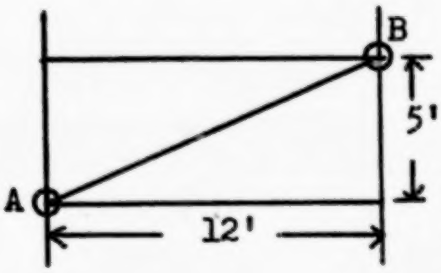
$$\begin{array}{r} 17 \\ \times\ 17 \\ \hline 119 \\ 17\phantom{0} \\ \hline 289 \end{array}$$

4. Practice

Find answers to nearest whole number

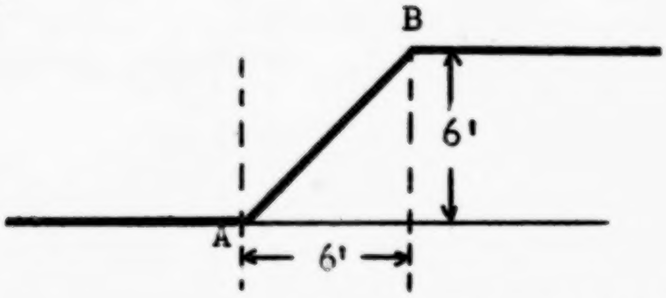
a. On a blueprint for a new building, an electrician's foreman sees these dimensions:

What is the length of conduit needed from A to B?



b. A 4" drain pipe is shown on a blueprint as follows:

How much pipe is needed between A and B?



c. A man is building a garage 14' by 20'. If the sides of the garage are at right angles, what is the measurement diagonally across the garage?

d. A tree is braced with a wire that is attached 8 feet up on the trunk, and to a stake in the ground 5 feet away, how long is the wire?

# VIII

# CONTEMPORARY MATHEMATICS - OTHER NUMBER BASES

## I. Expressing Number Ideas in the Decimal System

### A. Suggested Procedure

1. Help the pupil to develop the concept of one-to one correspondence through familiar situations.

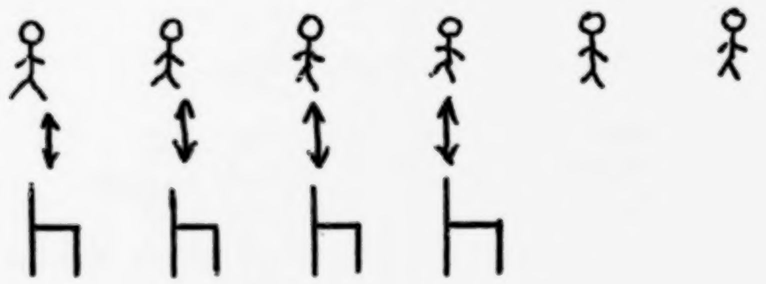
    a. Our class has 35 pupils. The office sent a set of mathematics books. Each pupil received a book. There were just enough books.

    This matching or pairing of a pupil with a book, or or a book with a pupil, is called one-to-one correspondence.

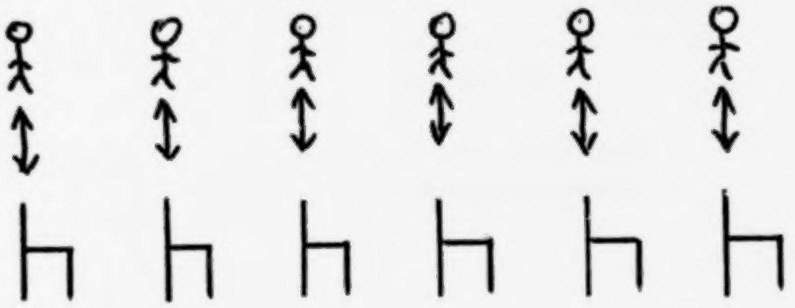
    b. Have the pupil use a situation which can be diagrammed.

    Six people went into a restaurant and were assigned to a table. There were only four chairs around the table. Elicit from the pupil that there is no one-to-one correspondence in this situation.

    Have pupil make a diagram of this situation



The waiter brought 2 more chairs

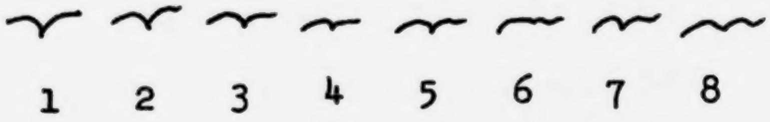


We now have 6 people and 6 chairs, a chair for each person and a person for each chair. There is now a one-to-one correspondence between the people and the chairs.

c. Have pupil diagram another situation from those previously suggested by the class.

John owned a flock of pigeons. Each day before setting them loose, he counted them.

Have pupil diagram this situation.



Tell why the above is a one-to-one correspondence.

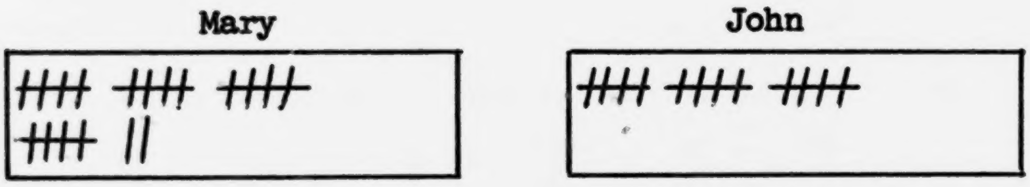
What was John matching?

"1" represents 1 pigeon
"2" represents 1 + 1 or a group of 2 pigeons.
"3" represents 2 + 1 or a group of 3 " , etc.

2. Help the pupil develop the concept of one-to-one correspondence through tallying.

a. Pose a problem such as:

Mary and John were candidates for class president. There were 37 pupils on register and all were present. Each pupil present cast a vote. Show by the use of tallies the number of votes received by John and by Mary.



22 votes
15 votes

Explain how tallying is an illustration of one-to-one correspondence.

b. Practice

1) Select from each of the following pairs those which illustrate one-to-one correspondence.

a) 6 girls
b) 4 pairs of skates

c) 5 windows
d) 7 pairs of curtains

e) 8 dimes
f) 8 nickels

g) 25 books
h) 27 pupils

i) 5 shirts
j) 5 neckties

k) 2 pairs of sox
l) 4 feet

2) Tell whether the following are in one-to-one correspondence.

a) the number of vowels in our alphabet (a,e,i,o,u)
b) the usual school days in a week.

a) the four seasons
b) the days in the Thanksgiving holiday

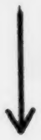
a) the days in April
b) the days in June

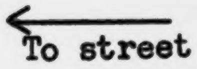
a) the angles in a triangle
b) the angles in a rectangle

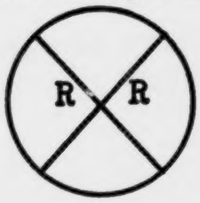
a) the stars in our American flag
b) the states in the United States

3. The numerals we have used in discussing one-to-one correspondence represent ideas. In our everyday activities, we frequently use symbols to represent ideas.

a. Have pupil suggest some non-mathematical symbols such as:







Have pupil interpret each symbol

b. Have pupil recall and suggest some mathematical symbols.

Some likely suggestions

1) Basic number symbols

2) Symbols of operation: +, -, ÷, x.

3) Equality sign =

B. Suggested Practice

Use diagrams to illustrate the situations presented in 2-b-1).

Note to Teacher: Since the words "number" and "numeral" will appear many times in the text that follows, it is suggested that the teacher review with the pupils

1) number refers to the abstract idea, e.g., "threeness"

2) any representation of number is called a numeral, e.g., "3", "III".

However, when we are sure that no confusion will result, we will use the symbol to mean either the number or the numeral.

II. Place Value - Decimal System

A. Suggested Procedure

Note: We will use the following language with the students:

There are ten basic symbols or digits for expressing whole numbers. A numeral, such as 4, is a digit, or a group of digits such as 357. The numeral 357 is made up of the basic symbols or digits 3, 5, and 7. If a number can be represented by a numeral consisting of two digits such as 46, we will call the number a 2-digit number. The number 357 is a 3-digit number.

1. Have pupil review 10 basic number symbols.

a. Have them suggest some 1, 2, 3, and 4-place numbers.

b. Have them realize that any of these numbers are expressed by the use of the 10 basic number symbols.

c. Have the pupil realize that numbers of any size may be expressed with one or more of the basic number symbols.

2. Help pupil develop a counting table to reinforce the meaning of base 10 and place value.

   a. Pupil knows that we use one-digit numerals to represent each number through 9.

   b. Help pupil see that as we reach each ten in counting, we group the ten ones into a single group ////////// called 1 ten; then combine the one ten with the number of tens in tens place.

Suggested Counting Table

| | Symbol | Meaning |
|---|---|---|
| | 0 | no ones |
| | 1 | 1 one |
| | 2 | 2 ones |
| | 3 | 3 ones |
| | 4 | 4 ones |
| | 5 | 5 ones |
| | 6 | 6 ones |
| | 7 | 7 ones |
| | 8 | 8 ones |
| | 9 | 9 ones |
| 9 ones + 1 one = 10 ones = 1 ten | 10 | 1 ten and no ones |
| | 11 | 1 ten and "1" one |
| | 12 | 1 ten and "2" ones |
| | 19 ......... | 1 ten and "9" ones |
| 19 + 1 = 1 ten and 10 ones; 10 ones = 1 ten; 1 ten + 1 ten = | 20 | 2 tens and no ones |
| | 21 | 2 tens and "1" ones |
| | 22 | 2 tens and "2" ones .. |
| | 29 | 2 tens and "9" ones |
| 29 + 1 = 2 tens and 10 ones = 2 tens + 1 ten = | 30 | 3 tens and no ones |
| | 31 | 3 tens and "1" one |
| | 32 ......... | 3 tens and "2" ones |
| | 39 | 3 tens and "9" ones |

|  |  |  |
|---|---|---|
| 39 + 1 = 3 tens + 10 ones = 3 tens + 1 ten = | 40 ...... | 4 tens and no ones |
|  | 90 | 9 tens and no ones |
|  | 91 | 9 tens and "1" one |
|  | 92 | 9 tens and "2" ones |
|  | 99 | 9 tens and "9" ones |
| 99 + 1 = 9 tens + 10 ones = 9 tens + 1 ten = 10 tens = 10 x 10 = 100 | 100 | 10 tens and no ones or 100 |

c. Have the pupil observe the pattern. He sees that:

1) We group by 10's, e.g., 10, 20, 30, 40, .... 100, etc.

2) Each time we reach ten ones we group the ten ones into one ten. Then the tens place is increased by one.

3) Tell the pupil that since we use ten basic number symbols and group by tens our system is called a base ten system or a decimal system. The word decimal means "ten".

4) Have the pupil realize that:

20 is 2 times the base ( 2 x 10)
30 is 3 times the base ( 3 x 10)
40 is 4 times the base ( 4 x 10)
100 is 10 times the base (10 x 10 or 100)

d. Practice

Fill in the blanks with the correct word or words:

1) The ten basic number symbols are________________.

2) Using combinations of the basic number symbols, we can write numbers___________________.

3) Our number system is called ___________ or the decimal system.

4) Using the symbols 4, 3, 9 write:

a) the largest numeral

b) the smallest numeral

5) Using the number symbols 3, 8, 0, 4 write:

a) the largest 4 place number
b) the smallest 4 place number

e. Through the use of a place value chart have the pupil realize the importance of the concept of place value and the use of zero.

Place Value Chart

| Thousands | Hundreds | Tens | Ones |
|---|---|---|---|
| 10 x 10 x 10 | 10 x 10 | 10 | 1 |
|  |  | 8 | 8 |
|  | 2 | 2 | 2 |
|  | 9 | 9 | 9 |
|  | 7 | 0 | 7 |
| 2 | 0 | 4 | 0 |
| 9 | 0 | 0 | 0 |

1) Have pupil reinforce the concept that the base value of each digit in a numeral is ten times as great as the digit immediately to its right. Thus in 88, 8 tens is ten times the value of 8 ones. In 999, 9 tens is 10 times the value of 9 ones. 9 hundreds is ten times the value of 9 tens.

2) Have pupil generalize:
As we move from the right to the left each digit is ten times as great as the digit immediately to the right; e.g., in 444, the 4 in tens place is ten times the value of 4 in the ones place.

3) Have pupil give the value of each digit in the last three numerals on the chart; e.g.,

7 0 7 means:
"7" ones
"0" tens
"7" hundreds

4) Help the pupil realize the importance of zero in our decimal system. In the numeral 707, the zero shows there are no tens. In 4085, the zero shows

there are no hundreds. Zero is used in numerals to show that there are no ones, no tens, no hundreds, and so on.

5) Have the pupil expand each numeral on the chart.

88 = 8 tens + 8 ones
= (8 x 10) + (8 x 1)
= 80 + 8

237 = (2 x 100) + (3 x 10) + (7 x 1)
= (2 x 10 x 10) + (3 x 10) +(7 x 1)
= (200) + (30) + 7

999 = (9 x 100) + (9 x 10) + (9 x 1) etc.
= (900) + (90) + (9)

3. Have the pupil use exponents in writing expanded numerals.

a. Review the meaning of:

1) exponent

2) the value of each of the following:

a) $3^2$; $4^2$; $7^2$
b) $2^3$; $3^3$; $5^3$
c) $10^2$; $10^3$

b. Expand 387 using exponents.

387 means (3 x 100) + (8 x 10) + (7 x 1)
=(3 x $10^2$) + (8 x 10) + 7

2496 means (2 x 1000) + (4 x 100) + (9 x 10) + (6 x 1)
=(2 x $10^3$) + (4 x $10^2$) + (9 x 10) + 6

c. Practice

Expand each of the following using exponents.

1) 189 2) 346 3) 792 4) 1288 5) 8,517

4. Sum up the important facts to remember about the decimal system.

a. We use 10 basic number symbols (0 through 9).

b. We group by tens. (Base 10 and place value)

c. Our numeral system is a decimal system because it has a base of ten.

d. Each digit in a numeral has a value depending upon its place in the numeral.

e. The symbol and its place value determine the value of the number, e.g., 85 means 8 tens and 5 ones.

5. Help pupil construct a place value table.

Place Value Table

| Thousands | Hundreds | Tens | Ones |
|---|---|---|---|
| Base x Base x Base<br>1000<br>10x10x10 | Base x Base<br>100<br>10x10 | Base<br>10 | 1 |
| $10^3$ | $10^2$ | 10 | 1 |

Have the pupil use the table to write expanded numerals

B. Suggested Practice

1. Fill in the blanks with the correct answer.

a. The numeral system has______ basic symbols. These include:
__________.

b. In the decimal system we group by _____________.

c. 17 means 1___________ and 7______________.

d. In $2^3$, the exponent is________.

e. $2^3$ = __________.

2. Write these numerals using exponents:

   100 50 300 2000

3. a. Write the largest possible number using the digits 6, 9, 3.

   b. Write the smallest possible number using the digits 3, 1, 6.

4. Expand each of the following numerals using exponents:

   a. 7283 b. 5086 c. 7800 d. 4304 e. 6759

5. If you write 6 and place a zero after it, what is the value of the numeral? By what number has 6 been multiplied?

6. How, in our number system is zero used to tell how many? Explain.

7. Give the meaning of the powers below.

   Example: $5^3 = 5 \times 5 \times 5$

   a. $10^3$ e. $10^1$

   b. $3^2$ f. $5^1$

   c. $7^3$ g. $4^2$

   d. $10^2$ h. $6^3$

8. What is the place value of the 7 in each of the following:

   a. $4\underline{7}02$ b. $42\underline{7}0$ c. $\underline{7}300$ d. $650\underline{7}$

9. a. Write the largest one place number.

   b. Write the largest two place number.

   c. Write the largest three place number.

   d. Write the smallest one place counting number

   e. Write the smallest two place counting number.

   f. Write the smallest three place counting number

10. Write the consecutive number that follows each of the following:

    a. 9 b. 99 c. 999

## II. Base Five Numeration System - Meaning

### A. Introductory Discussion

1. Have pupil suggest possible reasons for our numeral system having been based on the number ten (ten fingers).

2. Tell pupils that people have thought in terms of other number bases, e.g., twenty. The Bible speaks of a man's life span as being "three score and ten"; Lincoln spoke of our country's having been established "four score and seven years ago".

### B. The concept of numerals in base five

1. Have pupils group discs in tens and in fives.

a. 11 discs

| By tens | Tens | Ones | Fives | Ones |
|---|---|---|---|---|
| $\boxed{x\ x\ x\ x\ x\ x\ x\ x\ x\ x}$ x | 1 | 1 | | |
| By fives | | | | |
| $\boxed{x\ x\ x\ x\ x}$ $\boxed{x\ x\ x\ x\ x}$ x | | | 2 | 1 |

b. 6 discs

| | | | | |
|---|---|---|---|---|
| By tens | | | | |
| x x x x x x | | 6 | | |
| By fives | | | | |
| $\boxed{x\ x\ x\ x\ x}$ x | | | 1 | 1 |

c. 10 discs

| | | | | |
|---|---|---|---|---|
| By tens | | | | |
| $\boxed{x\ x\ x\ x\ x\ x\ x\ x\ x\ x}$ | 1 | 0 | | |

| | Tens | Ones | Fives | Ones |
|---|---|---|---|---|
| By fives | | | | |
| $\boxed{x\ x\ x\ x\ x}$ $\boxed{x\ x\ x\ x\ x}$ | | | 2 | 0 |

d. 15 discs

| | Tens | Ones | Fives | Ones |
|---|---|---|---|---|
| By tens | | | | |
| $\boxed{x\ x\ x\ x\ x\ x\ x\ x\ x\ x}$ x x x x x | 1 | 5 | | |
| By fives | | | | |
| $\boxed{x\ x\ x\ x\ x}$ $\boxed{x\ x\ x\ x\ x}$ $\boxed{x\ x\ x\ x\ x}$ | | | 3 | 0 |

2. Have pupil compare the way in which the numerals are written in each grouping.

   a. Arrange the numerals in columns.

| | Column I (No. of discs in groups of tens) | Column II (No. of discs in groups of fives) |
|---|---|---|
| 1. | 6 | Fives One<br>↘ ↙<br>11 |
| 2. | 11 | 21 |
| 3. | 10 | 20 |
| 4. | 15 | 30 |

   b. In what way are the numbers in the columns alike?

   Elicit from pupils that (from column I above) the numerals on each line (across) of Column I and Column II represent the same number of discs.

c. In what way are the numerals in the columns different? Elicit from pupil that the numerals representing the number of discs (on each line across) are different. This is so because the numerals in Column I represent groupings by ten, i.e. are written in base ten; the numerals in Column II represent groupings by five, i.e. are written in base five.

d. Tell pupils that there is a system of numerals in the base five. It is called the quinary or base five numeration system.

e. Reading numbers in base five.

   1) Have pupils realize that 1 five and 1 one (as in 2a) cannot be read eleven because eleven means 1 ten and 1 one. Therefore it must be read in some other way.

   2) Have pupil learn to read Column II in 2a as

      one one, base five
      two one, base five
      two zero, base five
      three zero, base five

f. Practice

   1) Encircle each of the following in groups of ten. Write the proper numeral in each case.

      a) x x x x x x x

      b) x x x x x x x x x x x

      c) x x x x x x x x x x x x x x x x x x x

      d) x x x x x x x x x x x x x x x x x x x x x

   2) Regroup the x's in 1) in fives, and write the proper numeral in each case.

C. The numerals in base five

1. Review

   a. Symbols or digits

   b. Number of symbols used in base ten (10).

c. Place value in base ten

Elicit

| | |
|---|---|
| 1) Largest one-digit number | 9 |
| Next number is base number | 10 |
| 2) Largest two-digit number | 99 |
| Next number is | 100 |
| 3) Largest three-digit number | 999 |
| Next number is | 1000 |

d. Have pupil observe that in each of the cases he has extended his place value table one place to the left.

2. Elements of the quinary numeration system

   a. Elicit from pupil that in base five there are five digits or symbols 0, 1, 2, 3, 4.

   b. Have him recall that in the decimal system the number following the largest one place number 9, is the base number ten, written 10 and meaning 1 ten and 0 ones.

   c. Have him see, similarly, that the largest one place number in base five is 4 and that the number following 4 is the base number five, written 10 and meaning 1 five and 0 ones.

3. Help pupil list the five symbols in base five consecutively.

   a. The Ones Place

      0
      1
      2
      3
      4

   b. The Fives Place

      1) Have pupils recall (B1) that the column to the left of ones column is the fives column.

| ase Ten | Base Five | Read (in base five) | Values in base five |
|---|---|---|---|
| 0 | 0 | zero | no ones |
| 1 | 1 | one | one one |
| 2 | 2 | two | two ones |
| 3 | 3 | three | three ones |
| 4 | 4 | four | four ones |
|  |  | 2)<br>then five will be represented by a 1 in the fives column and zero in the ones column since five and zero are five. |  |
| 5 | 10 | one zero | one five, no ones |
| 6 | 11 | one one | one five, one one |
| 7 | 12 | one two | one five, two ones |
| 8 | 13 | one three | one five, three ones |
| 9 | 14 | one four | one five, four ones |
|  |  | 3)<br>Elicit from pupils that as in base ten, the next number after 19 is represented by two times the base number (2 x 10; or 20); so in base five, the consecutive number after $14_{five}$ is two times the base number 5 or $20_{five}$ which represents two fives and no ones. |  |
| 10 | 20 | two zero | two fives, no ones |
| 11 | 21 | two one | two fives, one one |
| 12 | 22 | two two | two fives, two ones |
| 13 | 23 | two three | two fives, three ones |
| 14 | 24 | two four | two fives, four ones |
|  |  | 4)<br>Elicit the next number from a study of 2) and 3). |  |
| 15 | 30 | three zero | three fives, no ones |
| 19 | 34 | three four | three fives, four ones |

5\)
Have pupils study the four groups 1) , 2) , 3) , 4) , above.

Guide them to decide after what number there was a change in the fives column.

What follows $34_{five}$ ? (40 which means four fives, no ones)

c. The Twenty-fives Place

1) Have pupil consider what number will follow 44 in base five. Relate this to 99 in base ten.

2) Elicit from pupil: What happens in base 5 each time we count consecutively from a numeral ending in 4?

e.g. 4 five
+ 1 five
$10_{\text{five}}$ ( read, "one zero, base five")
value = one five, no ones)

44 five
+ 1
$100_{\text{five}}$ (read, "one zero zero, base five")

Lead pupil to conclude that when counting in base five, the sum of "4" and "1" result in a zero ("0") and an exchange of "1" to the next column to the left.

3) This third place represents the twenty-fives place or the 5 x 5 place or the $5^2$ place.

| Base Ten | Twenty-fives $5 \times 5$ or $5^2$ | Fives | Ones | Values |
|---|---|---|---|---|
| 25 | 1 | 0 | 0 | five squared no one's |

4) Help pupil continue listing the following numbers

| Base Ten | $5^2$ or 25 | fives | ones | Values |
|---|---|---|---|---|
| 26 | 1 | 0 | 1 | five squared and one |
| 27 | 1 | 0 | 2 | five squared and two |
| 29 | 1 | 0 | 4 | five squared and four |
| 30 | 1 | 1 | 0 | five squared and five |
| 35 | 1 | 2 | 0 | five squared and 2fives |
| 41 | 1 | 3 | 1 | five squared, 3fives and 1 |
| 49 | 1 | 4 | 4 | five squared, 4" and four |
| 50 | 2 | 0 | 0 | 2 times five squared |
| 51 | 2 | 0 | 1 | 2 times five squared and one |
| 69 | 2 | 3 | 4 | 2 times five squared 3 fives four |
| 75 | 3 | 0 | 0 | 3 times five squared |
| 100 | 4 | 0 | 0 | 4 times five squared |
| 124 | 4 | 4 | 4 | 4 times five squared, 4 fives and four |

d. Following the previous pattern, the pupil will recognize that at this point the system would have to be extended another place to the left.

Note: This place would be the 125 place or 5 x 5 x 5 place or $5^3$ place. Numbers beyond three places will not be treated in this development.

4. Tell pupils that when we wish to indicate that a number is written in a base other than ten, we name the base number (here five) as a subscript a little below and to the right of the number, e.g.

   $4_{\text{five}}$ $31_{\text{five}}$ $100_{\text{five}}$

   Note: The five written five not 5.

5. Practice

   a. Rewrite the following to indicate that they are numerals in base five:

      23 30 12 41

   b. Write in column II in base five the number that follows consecutively the one in column I

| Column I | Column II |
|---|---|
| $4_{\text{five}}$ | $10_{\text{five}}$ |
| $14_{\text{five}}$ | |
| $23_{\text{five}}$ | |
| $44_{\text{five}}$ | |
| $100_{\text{five}}$ | |
| $234_{\text{five}}$ | |

D. Changing a base five numeral to the equivalent base ten numeral

1. Review expansion of base ten numerals to clarify meaning of place value; e.g. $354 = (3 \times 10^2) + (5 \times 10^1) + (4 \times 1)$

2. Have pupils expand numerals in base five to re-enforce understanding of place value.

a. Expanding $342_{five} = (3 \times 5^2) + (4 \times 5) + (2 \times 1)$

$101_{five} = (1 \times 5^2) + (0 \times 5) + (1 \times 1)$

b. Practice

Expand: $240_{five}$ $100_{five}$ $444_{five}$

$44_{five}$ $131_{five}$

3. Changing a base five numeral to the equivalent base ten numeral.

a. Repeat expansion of base five numeral as in 2a.

1) $342_{five} = (3 \times 5^2) + (4 \times 5) + (2 \times 1)$

$= (75) + (20) + (2)$

$= 97_{ten}$ or 97

2) $101_{five} = (1 \times 5^2) + (0 \times 5) + (1 \times 1)$

$= 25 + 0 + 1 = 26_{ten}$ or 26

Question: Why does the second parenthesis yield a zero?

b. Practice

1) Expand the following numerals in base ten:

24 99 310 205 100

2) Expand the following numerals in base five:

$10_{five}$ $14_{five}$ $30_{five}$ $124_{five}$ $200_{five}$

3) Find the value in base ten of the base five numerals in 2).

Note: When a subscript is omitted, the base 10 is understood.

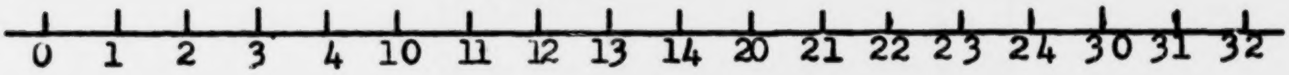
## IV. Base Five - Operations

### A. The number line

1. Construction of a number line

   a. Have the pupils construct a number line using the numerals in base five.

   b. The resulting number line should be:



   c. Have the pupils realize that each number to the right is one more than the preceding number. In base five, 1 more than 4 is $10_{five}$.

   Explain.

2. Addition on the number line.

   a. Problem: What is the sum of $4_{five}$ and $3_{five}$?

   b. Ask the pupils how the number line could be used to find this sum. Have him find 4 on the number line. Have pupil move his finger 3 spaces to the right. His finger should now be resting on $12_{five}$. Thus, the sum of $4_{five}$ and $3_{five}$ is $12_{five}$.

   c. Practice:

   Have the pupil use the base five number line to find the sum of the following (base five):

   1 $\begin{array}{r} 3 \\ +1 \\ \hline \end{array}$ 2 $\begin{array}{r} 4 \\ +2 \\ \hline \end{array}$ 3 $\begin{array}{r} 11_{five} \\ +\ 3_{five} \\ \hline \end{array}$ 4 $\begin{array}{r} 3_{five} \\ +\ 3_{five} \\ \hline \end{array}$ 5 $\begin{array}{r} 0_{five} \\ +11_{five} \\ \hline \end{array}$

3. Developing a table of addition facts base five:

a. Help pupil to develop a table of addition facts in base 5.

BASE FIVE - ADDITION TABLE

| + | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 |  |  |
| 1 |  |  |  |  |  |
| 2 |  |  |  |  |  |
| 3 |  |  |  |  |  |
| 4 |  |  |  |  |  |

Suggestion: Copies to be supplied to pupils for completion during the lesson.

b. What is placed in each space? Elicit from the pupils the fact that each space contains the sum of two numbers. One of the addends is at the left and one of the addends is at the top. 0 + 0 = 0. In the space represented by 0 + 0 write a 0. 0 + 1 = 1. In the space for 0 + 1 write 1. In the same way, fill in all the empty spaces. Use the number line where necessary to find the sums. The resulting table should be:

BASE FIVE -ADDITION TABLE

| + | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 1 | 2 | 3 | 4 | 10 |
| 2 | 2 | 3 | 4 | 10 | 11 |
| 3 | 3 | 4 | 10 | 11 | 12 |
| 4 | 4 | 10 | 11 | 12 | 13 |

c. Elicit from pupil:

The table eliminates the need for using the number line to find the sum of two numbers base five.

B. Addition of one place numbers in base five.

1. Have pupil realize that he will need to use the table to add because he has not memorized the addition facts in base five as he has in base ten.

2. Problem: What is the sum of $2_{five} + 4_{five}$ ?

   In the base five addition table,
   find 2 at the extreme left;
   then find 4 at the very top;
   then find the space associated with these two numbers.

   The number 11 is in this space. $11_{five}$ is

   the sum of $2_{five} + 4_{five}$

3. Practice: Use the addition table of base five to find to find the following sums:

   a) $\begin{array}{r} 4 \\ +\ 0 \\ \hline \end{array}$ b) $\begin{array}{r} 2 \\ +\ 2 \\ \hline \end{array}$ c) $\begin{array}{r} 2 \\ +\ 3 \\ \hline \end{array}$ d) $\begin{array}{r} 3 \\ +\ 2 \\ \hline \end{array}$ e) $\begin{array}{r} 3 \\ +\ 4 \\ \hline \end{array}$

   Note: Answers must be labeled "base five".

C. Addition of two place numbers base five (no exchange)

1. Problem: Add $\begin{array}{r} 23 \\ +\ 21 \\ \hline \end{array}$

   a. Find the sum of 3 + 1 in the base five table. It is 4.

   b. Write 4 in ones column.

   c. Find the sum of 2 + 2 in the table. It is 4.

   d. Write the 4 in fives column.

   The sum of $\begin{array}{r} 23 \\ +\ 21 \\ \hline \end{array}$ is $44_{five}$.

2. Practice: Add in base five.

   a) $\begin{array}{r} 12 \\ +\ 10 \\ \hline \end{array}$ b) $\begin{array}{r} 13 \\ +\ 11 \\ \hline \end{array}$ c) $\begin{array}{r} 14 \\ +\ 10 \\ \hline \end{array}$ d) $\begin{array}{r} 11 \\ +\ 3 \\ \hline \end{array}$ e) $\begin{array}{r} 20 \\ +\ 20 \\ \hline \end{array}$ f) $\begin{array}{r} 10 \\ +\ 4 \\ \hline \end{array}$

   Label answers "base five" as above B,3,a.

D. Addition involving exchange:

1. Have the pupils review addition with exchange in the decimal numeral system.

   Problem: Add in base ten.

   $$\begin{array}{r} 26 \\ +\ 37 \\ \hline \end{array}$$

   a. The 6 and the 7 are in ones place. The 2 and the 3 are in tens place.

   b. $6 + 7 = 13$ or 1 ten and 3.

   c. Write the 3 in ones place.

   d. Add 1 ten to the numbers in tens place. The sum is 6.

   e. Write the 6 in tens place.

   f. The sum is 63.

2. Use the same method to add where exchange is involved in base five. (Use table where necessary.)

   Problem: Add in base five.

   $$\begin{array}{r} 33 \\ +\ 24 \\ \hline \end{array}$$

   a. Add ones column: $3 + 4 = 12_{\text{five}}$.

   b. Write the 2 in ones place.

   c. Add 1 five to the number in fives place.

   d. The sum of $3 + 2 + 1$ in fives place is $11_{\text{five}}$.

   e. The sum of $\begin{array}{r} 33 \\ +\ 24 \\ \hline \end{array}$

      is $112_{\text{five}}$

3. Practice: Find the sum of the following in base five. Use the base five addition table where necessary.

$$\text{a)}\ \begin{array}{r} 23 \\ +\ 32 \\ \hline \end{array} \qquad \text{b)}\ \begin{array}{r} 43 \\ +\ 12 \\ \hline \end{array} \qquad \text{c)}\ \begin{array}{r} 24 \\ +\ 4 \\ \hline \end{array} \qquad \text{d)}\ \begin{array}{r} 43 \\ +\ 3 \\ \hline \end{array} \qquad \text{e)}\ \begin{array}{r} 22 \\ +\ 13 \\ \hline \end{array}$$

All sums must carry subscripts indicating the base.

E. Subtraction in base five.

1. Review concept of subtraction in base ten.

   a. Show that it is possible to find the answers in a subtraction example by an addition process.

$$\begin{array}{r} 6 \\ +\ 7 \\ \hline 13 \end{array} \qquad\qquad \begin{array}{r} 6 \\ +\ ? \\ \hline 13 \end{array} \qquad\qquad \begin{array}{r} 7 \\ +\ ? \\ \hline 13 \end{array}$$

   b. Express the above as subtraction examples

   $13 - 6 = 7$ ;  $13 - 7 = 6$

   c. Practice:

      1) Indicate:

      What must be added in each case to give the sum?

$$\text{a)}\ \begin{array}{r} 4 \\ +\ ? \\ \hline 9 \end{array} \qquad \text{b)}\ \begin{array}{r} 8 \\ +\ ? \\ \hline 16 \end{array} \qquad \text{c)}\ \begin{array}{r} 9 \\ +\ ? \\ \hline 15 \end{array} \qquad \text{d)}\ \begin{array}{r} 7 \\ +\ ? \\ \hline 13 \end{array} \qquad \text{e)}\ \begin{array}{r} 5 \\ +\ ? \\ \hline 15 \end{array}$$

      2) Express each of the above as a subtraction example.

$$\text{a)}\ \begin{array}{r} 9 \\ -\ 4 \\ \hline \end{array}$$

2. Help the pupil: subtract in base five (Use the table).

   a. Problem: $12 - 4 = ?$

      1) Have the pupil find 4 at the extreme left of the table, and place one finger on it.

      2) Tell him to move the finger to the right until 12 is reached.

      3) Then have him move his finger to the top of this column.

      4) The finger should rest on 3.

5) $12_{five} - 4_{five} = 3_{five}$ or $\begin{array}{r} 12_{five} \\ -\ 4_{five} \\ \hline 3_{five} \end{array}$

b. Checking a subtraction example

1) Checking in the decimal system is done by addition. In base ten, $13 - 9 = 4$. As a check, add $9 + 4$. The sum is 13. Thus, addition was used to check the accuracy of the calculations in subtraction.

2) We have found that in base five $12 - 4 = 3$.

The addition example thus checks the subtraction example.

3. Practice:

Subtract using the base five addition table. Check by addition.

a) $\begin{array}{r} 10 \\ -\ 1 \\ \hline \end{array}$ b) $\begin{array}{r} 3 \\ -\ 2 \\ \hline \end{array}$ c) $\begin{array}{r} 11 \\ -\ 4 \\ \hline \end{array}$ d) $\begin{array}{r} 12 \\ -\ 3 \\ \hline \end{array}$ e) $\begin{array}{r} 20 \\ -\ 10 \\ \hline \end{array}$

F. Subtraction involving exchange (Optional)

1. Subtract $\begin{array}{r} 23 \\ -\ 15 \\ \hline \end{array}$ in the decimal system.

Have pupils recall that they exchanged as follows:

$\begin{array}{rrr} & 1 & 13 \\ & \cancel{2} & \cancel{3} \\ - & 1 & 5 \\ \hline & & 8 \end{array}$ because 1 ten equals 10 ones.

2. Subtraction base five involving exchange (use table)

a. Problem: Subtract $\begin{array}{r} 22 \\ -\ 14 \\ \hline \end{array}$ in base five.

1) The same technique of exchange as used in the decimal system can be used in subtraction base five.

2) $\begin{array}{r} 22 \\ -\ 14 \\ \hline \end{array}$ becomes $\begin{array}{rrr} & 1 & 12 \\ & \cancel{2} & \cancel{2} \\ - & 1 & 4 \\ \hline \end{array}$

because 1 five equals 5 ones.

3) Use table to find the difference between 12 and 4. The difference is $3_{five}$.

b. Practice: Find the differences in the following subtraction examples base five. Check by addition.

$$1)\ \begin{array}{r} 43 \\ -\ 24 \\ \hline \end{array} \qquad 2)\ \begin{array}{r} 32 \\ -\ 14 \\ \hline \end{array} \qquad 3)\ \begin{array}{r} 41 \\ -\ 2 \\ \hline \end{array} \qquad 4)\ \begin{array}{r} 41 \\ -\ 4 \\ \hline \end{array}$$

All answers must be marked with subscript to indicate the base.

## V. Binary System

### A. Motivating Discussion

1. Trick: Find the missing number

| 16 | 8 | 4 | 2 | 1 |
|---|---|---|---|---|
| 17 | 9 | 5 | 3 | 3 |
| 18 | 10 | 6 | 6 | 5 |
| 19 | 11 | 7 | 7 | 7 |
| 20 | 12 | 12 | 10 | 9 |
| 21 | 13 | 13 | 11 | 11 |
| 22 | 14 | 14 | 14 | 13 |
| 23 | 15 | 15 | 15 | 15 |
| 24 | 24 | 20 | 18 | 17 |
| 25 | 25 | 21 | 19 | 19 |
| 26 | 26 | 22 | 22 | 21 |
| 27 | 27 | 23 | 23 | 23 |
| 28 | 28 | 28 | 26 | 25 |
| 29 | 29 | 29 | 27 | 27 |
| 30 | 30 | 30 | 30 | 29 |
| 31 | 31 | 31 | 31 | 31 |

This table may be rexographed in advance, and a copy given to each pupil.

2. Send a pupil to the blackboard. (The teacher should face the class so that the chart and the number are out of his sight. Tell the pupil to write any number from 1 through 31. Have the class answer the following questions.

   a. Is the number in the first column? (Going from right to left.)

   b. Is it in the second column?

   c. Is it in the third column?

   d. Is it in the fourth column?

   e. Is it in the fifth column?

3. The teacher will then guess the number. This is done by getting the sum of the numbers at the head of the columns in which the number to be guessed appears.

   a. Example: Suppose the number written by the pupil is 13. This number appears in the columns and is found as follows.

| No. | Columns in Which It Appears | No. at Head of Column |
|---|---|---|
| 13 | 1st | 1 |
|  | 3rd | 4 |
|  | 4th | 8 |
|  |  | Sum 13 = No. |

4. Elicit the method used for guessing the number. If no pupil can give the method, the teacher should explain it.

5. Have pupils "try it out" on other members of the class.

6. Pupils will be interested to learn how the table was constructed. The method follows:

   a. Set up columns. Head these 1, 2, 4, 8, 16, going from right to left.

| 16 | 8 | 4 | 2 | 1 |
|---|---|---|---|---|

   b. 1 and 2 are in the table. 3 is the sum of 2 and 1 and should be written in the columns headed by 2 and 1. (See Table in A. 1.)

   c. 4 is in the table.

   d. 5 is the sum of 4 and 1 and should be written in the columns headed by 4 and 1.

   e. 6 is the sum of 4 and 2 and should be written in the columns headed by 4 and 2.

   f. Continue this procedure until the numbers through 31 are written in the appropriate columns. Example: 23 is the sum of 16, 4, 2, and 1. Hence 23 is written in the columns headed by 16, 4, 2, and 1.

**Note:** The numbers other than 1, 2, 4, 8, and 16 must be expressed as a sum of a combination of two or more of these numbers.

g. Pupils should be expected to complete the table after the teacher's demonstration of the general method for numbers 1 through 9.

7. Making use of the table in A.1. and going from left to right, pupils are led to write a given number symbolically on the basis of the columns in which the number appears. Thus, if the number appears in a column, 1 is written in that column. If it does not, 0 is written in that column. Thus, 13 appears in the column headed 8. Then 1 is written in this column. It does appear in the column headed 4, hence 1 is written in this column. It does not appear in in the column headed 2, hence 0 is written in this column. It also appears in the column headed 1, hence 1 is written in this column. This is illustrated below.

| 16 | 8 | 4 | 2 | 1 |
|---|---|---|---|---|
|  | 1 | 1 | 0 | 1 |

The number expressed symbolically is 1101.

**Note:** Pupils should be instructed to begin with the column (going from left to right) in which the number first appears.

8. Suggested Practice.

Making use of the table in A.1. and the symbols 1 and 0, write the numbers 1 through 31 symbolically.

B. Establishing place value in Base Two

1. Consider the table in A. 1. It will be seen that the number at the head of each column, beginning with 2, is twice the number at the head of the column immediately to its right. Thus, 2 is 2 x 1; 4 is 2 x 2; 8 is 2 x 4 = 2 x 2 x 2; and 16 is 2 x 8 = 2 x 2 x 2 x 2. In each of these illustrations the factor 2 is called the base. The number of times 2 is taken as a factor is written to the right of and a little above the base 2. Thus 8 = 2 x 2 x 2 = $2^3$; 16 = 2 x 2 x 2 x 2 x $2^4$. The number of times 2 is taken as a factor is called the exponent of 2. Thus, in $2^4$, 4 is the exponent of the base 2.

Elicit the fact that 2 as a base without an exponent implies 2 with exponent 1; i.e., $2 = 2^1$.

2. Write each of the following numbers in exponential form using 2 as the base.

   a. 4 b. 32 c. 128 d. 64

3. Write each of the following numbers in exponential form using 5 as a base.

   a. 5 b. 25 c. 625 d. 125

4. Write each of the following numbers in exponential form using 10 as a base.

   a. 10 b. 1000 c. 100 d. 10,000

5. Examine the table below and answer the questions that follow.

| Number | 64 | 32 | 16 | 8 | 4 | 2 | ? |
|---|---|---|---|---|---|---|---|
| Exponential Form | $2^6$ | $2^5$ | $2^4$ | $2^3$ | $2^2$ | $2^1$ | ? |

   a. How is each number obtained from the number immediately to its left?

   b. What should be the number in the table immediately to the right of 2?

   c. How does the exponent of 2 in each number compare with the exponent of 2 of the number immediately to its left?

   d. 1 may be written as 2 with what exponent?

   e. What is the value of $2^0$? ($2^0 = 1$)

6. Following the procedure of 5, above, develop the facts that $10^0$, $5^0$, $3^0$, all have the same value, namely, 1.

   The generalization should be elicited that any number (except 0) with exponent 0 is 1.

7. Rewrite the numbers heading the columns in Table A. 1., and above each write them in exponential form as follows:

| $2^4$<br>(16) | $2^3$<br>(8) | $2^2$<br>(4) | $2^1$<br>(2) | $2^0$<br>(1) |
|---|---|---|---|---|
| | | | | |

Using the table in B, 7, as a guide, have the pupils fill in the symbolic representations of numbers 1 through 31 in the base two. For example, 21 may be written in base two symbolically as in the table below.

| $2^4$<br>(16) | $2^3$<br>(8) | $2^2$<br>(4) | $2^1$<br>(2) | $2^0$<br>(1) |
|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 |

Conversely, given the symbolic representation of a number, develop the method for writing the number in base 10, as follows: $10101_{\text{two}} = ?$

1 in the $2^4$ column $= 2^4 = 2 \times 2 \times 2 \times 2 = 16$

1 in the $2^2$ column $= 2^2 = 2 \times 2 \quad = 4$

1 in the $2^0$ column $= 2^0 = 1 \times 1 \quad = 1$

$10101 = \quad 21$

The symbols 1 and 0 will be recognized as part of our decimal system. Pupils should also recognize that the symbols 1 and 0 may be considered numerals in this new number system. Since the columns used in this new system are headed $2^0$, $2^1$, $2^2$, etc., we may say that this system uses 2 as a base. The system is called the binary system of numbers. (Binary: Characterized by Two) Finally, we may also call numbers in this system Base Two Numbers. Base two uses two digits, zero and one.

(Enrichment) The digits used in base 10 are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The digits used in base five are 0, 1, 2, 3, 4. The digits used in base two are 0, 1.

Question: What digits are used in base 6? In base 7?
In base 8?

C. Writing and Reading Numerals 1 through 31 in Base Two

1. In expressing numbers in base two use the table in A. 1. as a guide and proceed as illustrated below.

| Number Base 10 | $2^4$ (16) | $2^3$ (8) | $2^2$ (4) | $2^1$ (2) | $2^0$ (1) | Number Base Two | How Read |
|---|---|---|---|---|---|---|---|
| 1 | | | | | 1 | $1_{two}$ | 1,base two |
| 2 | | | | 1 | 0 | $10_{two}$ | 1 0,base two |
| 3 | | | | 1 | 1 | $11_{two}$ | 1 1,base two |
| 4 | | | 1 | 0 | 0 | $100_{two}$ | 1 0 0,base two |
| 5 | | | 1 | 0 | 1 | $101_{two}$ | 1 0 1,base two |
| 6 | | | 1 | 1 | 0 | $110_{two}$ | 1 1 0,base two |
| 7 | | | 1 | 1 | 1 | $111_{two}$ | 1 1 1,base two |
| 8 | | 1 | 0 | 0 | 0 | $1000_{two}$ | 1 0 0 0,base two |
| 9 | | 1 | 0 | 0 | 1 | $1001_{two}$ | 1 0 0 1,base two |

Note: In writing a number in base two it is necessary to write "two" as a subscript.

2. Develop the pattern that the system of numeration in base two follows.

3. The pupils should complete table above through 31, base 10.

4. Pupils should be instructed as to how to read numerals base two. (See last column in table above.) Thus, $1001_{two}$ is read "one zero zero one ,base two."

D. Converting From Base Two to Base Ten (Limited to 1 through 31)

1. Pose the example: Change $11101_{two}$ to base 10.

2. Write 11101 in a table such as the one on the next page. (Insert the digits in the appropriate columns, going from right to left, and then fill in the last column.)

| Number Base Two | $2^4$ (16) | $2^3$ (8) | $2^2$ (4) | $2^1$ (2) | $2^0$ (1) | Synthesis (Sum in Expanded Form) | Number Base 10 |
|---|---|---|---|---|---|---|---|
| $11101_{two}$ | 1 | 1 | 1 | 0 | 1 | $2^4 + 2^3 + 2^2 + 2^0 =$<br>$16 + 8 + 4 + 1 =$ | 29 |

3. Suggested Practice: Change the following from base two to base ten:

a. $1111_{two}$ b. $10101_{two}$ c. $10000_{two}$

d. $10100_{two}$ e. $1110_{two}$

VI. Binary System - Operations

A. Addition in Base Two

1. Have pupil review counting in base 2, e.g., 1, 10, 11, 100, etc.

2. Have pupil review use of place value chart (through five places) to write base 2 numerals in base 10.

| Place Value Chart (Base Two) | | | | |
|---|---|---|---|---|
| Sixteens (24) | Eights (23) | Fours | Twos ($2^1$) | Ones ($2^0$) |
| 2 x 2 x 2 x 2<br>(16) | 2 x 2 x 2<br>(8) | 2 x 2<br>(4) | 1 x 2 | 1 |

For example, to express $1011_{two}$, in base 10 notation, pupils would consider the place value as follows:

$1011_{two}$ = 1 eight 0 fours 1 two 1 one = 11
that is 8 + 0 + 2 + 1
Then, $1011_{two}$ = $11_{ten}$

3. Addition Table in Base 2.

   a. Discuss with the pupil the use of an addition table in base 10 as a way of organizing addition facts for ready reference.

      **Note:** An addition fact is the combining of two one-digit numbers to arrive at a sum.

   b. Have pupil see that since a base 2 numeration system contains only two symbols, 0 and 1, there are but four addition facts.

      These are:

$$\begin{array}{r} 0 \\ +\ 0 \\ \hline 0 \end{array} \qquad \begin{array}{r} 0 \\ +\ 1 \\ \hline 1 \end{array} \qquad \begin{array}{r} 1 \\ +\ 0 \\ \hline 1 \end{array} \qquad \begin{array}{r} 1 \\ +\ 1 \\ \hline 10 \end{array}$$

      The addition facts for base 2 may be organized as follows:

| + | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 10 |

      To find the sum of two base 2 numbers by means of the table, e.g., to find the sum of 1 and 1, we locate 1 in the first column. It appears in the third row. We then locate 1 in the top row. It appears in the third column. Where the third row meets the third column, we find the sum 10.

   c. Have pupils practice adding numbers in base two, referring to the table of addition facts, if necessary.

      Example: Add 101 and 100 in binary notation.

      Pupil writes the addition example in the usual way:

$$\begin{array}{r} 101 \\ +\ 100 \\ \hline \end{array}$$

Starting at the right, we add 0 and 1. The result is 1. This is recorded in the answer in the columns of ones.

$$\begin{array}{r} 101 \\ +\ 100 \\ \hline 1 \end{array}$$

We then add 0 and 0 in twos column. The result is 0. We place 0 in the answer in two's column.

$$\begin{array}{r} 101 \\ +\ 100 \\ \hline 01 \end{array}$$

We next add 1 and 1 in the column of fours. This is 10 in binary notation. We write 0 in four's column and 1 in eight's column.

$$\begin{array}{r} 101 \\ +\ 100 \\ \hline 1001_{two} \end{array}$$

Have pupil check the answer by rewriting each numeral, as follows:

$$\begin{array}{rcc} \text{Base Two} & & \text{Base Ten} \\ 101_{two} & = & 5 \\ +\ 100_{two} & = & 4 \\ \hline 1001_{two} & = & 9 \end{array}$$

Then: Convert $1001_{two}$ to base ten and show

$$1001_{two} = 9_{ten}$$

4. Suggested Practice

Find the following sums in the binary notation. Check your answers by converting the sum in base two to base 10.

a. $\begin{array}{r} 101 \\ +\ 10 \\ \hline \end{array}$ b. $\begin{array}{r} 10 \\ +\ 10 \\ \hline \end{array}$ c. $\begin{array}{r} 11 \\ +\ 10 \\ \hline \end{array}$ d. $\begin{array}{r} 110 \\ +\ 10 \\ \hline \end{array}$ e. $\begin{array}{r} 11 \\ +\ 11 \\ \hline \end{array}$

Indicate sums in base two as $111_{two}$.

## B. Subtraction in Base 2

### 1. Without Exchange

a. Have pupil review subtraction without exchange in base 10.

$$\begin{array}{r} 9 \\ -\ 2 \\ \hline 7 \end{array} \qquad \begin{array}{r} 18 \\ -\ \ 6 \\ \hline 12 \end{array} \qquad \begin{array}{r} 27 \\ -\ 13 \\ \hline 14 \end{array} \qquad \begin{array}{r} 374 \\ -\ 212 \\ \hline 162 \end{array}$$

b. Have pupil subtract base 2 numbers in similar fashion.

$$\begin{array}{r} 1 \\ -\ 1 \\ \hline 0 \end{array} \qquad \begin{array}{r} 11 \\ -\ 1 \\ \hline 10 \end{array} \qquad \begin{array}{r} 110 \\ -\ 10 \\ \hline 100 \end{array}$$

Have pupil check the correctness of the computation by adding the remainder and the subtrahend.

### 2. With Exchange (Optional)

a. Have pupil review subtraction with exchange in base 10.

$$\begin{array}{r} 1\,15 \\ \cancel{2}\cancel{5} \\ -\ 18 \\ \hline 7 \end{array} \qquad \begin{array}{r} 7\,14 \\ \cancel{8}\cancel{4} \\ -\ 69 \\ \hline 15 \end{array} \qquad \begin{array}{r} 9\,12 \\ \cancel{1}\cancel{0}\cancel{2} \\ -\ \ \ 44 \\ \hline 58 \end{array}$$

Review with pupil the basic principle of exchange in base 10:

A unit in any place may be exchanged for ten (10) units of the place immediately to its right.

b. Have pupil utilize the principle of exchange in base 2:

$$\begin{array}{r} 0\,10 \\ 1\cancel{1}\cancel{0} \\ -\ \ \ 1 \\ \hline 101 \end{array} \qquad \begin{array}{r} 110 \\ \cancel{1}\cancel{0}\cancel{0} \\ -\ \ \ 1 \\ \hline 11 \end{array}$$

Have pupil check by adding the remainder and the subtrahend.

3. Suggested Practice

Find the following differences in binary notation and check.

a. $$\begin{array}{r} 101 \\ -\ 10 \\ \hline \end{array}$$

b. $$\begin{array}{r} 101 \\ -\ 11 \\ \hline \end{array}$$

c. $$\begin{array}{r} 100 \\ -\ 11 \\ \hline \end{array}$$

d. $$\begin{array}{r} 110 \\ -\ 11 \\ \hline \end{array}$$

C. Multiplication in Base 2

1. Discuss with pupil the use of a multiplication table in base 10 as a way of organizing multiplication facts for ready reference.

2. Guide pupil to see that in base 2 numeration system there are only four multiplication facts:

$$\begin{array}{r} 0 \\ \times\ 0 \\ \hline 0 \end{array} \qquad \begin{array}{r} 0 \\ \times\ 1 \\ \hline 0 \end{array} \qquad \begin{array}{r} 1 \\ \times\ 0 \\ \hline 0 \end{array} \qquad \begin{array}{r} 1 \\ \times\ 1 \\ \hline 1 \end{array}$$

The product is "0" in each case except for "1 x 1".

These facts may be organized in table form as follows:

| x | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

This table has been called "the schoolboy's delight". Can you tell why?

3. Have pupil note that in arranging numerals in multiplication, position is very important. For example, to multiply the binary numbers 10 by 11, we may place the numerals with the multiplier on the bottom. Then, as in ordinary multiplication, we begin by multiplying by the lower right-hand number.

Optional

Base Two

$$\begin{array}{r} 10 \\ \times\ 11 \\ \hline 10 \\ 10\phantom{0} \\ \hline 110_{two} \end{array}$$

Check

Base Two $\qquad\qquad\qquad\qquad$ Base Ten

$$\begin{array}{r} 10 \\ \times\ 11 \\ \hline 10 \\ 10\phantom{0} \\ \hline 110_{two} \end{array} \quad = \quad \begin{array}{r} 2 \\ \times\ 3 \\ \hline 6_{ten} \end{array}$$

$$\text{Does } 110_{two} = 6_{ten}$$

$$\begin{aligned} 110_{two} &= (1 \times 2^2) + (1 \times 2) + (0) \\ &= (1 \times 4) + (2) + (0) \\ &= 4 + 2 + 0 \\ &= 6_{ten} \end{aligned}$$

4. Suggested Practice

   a. Find the following products in binary notation and check your answer.

$$\begin{array}{r} 10 \\ \times 11 \\ \hline \end{array} \qquad \begin{array}{r} 100 \\ \times 10 \\ \hline \end{array} \qquad \begin{array}{r} 110 \\ \times 11 \\ \hline \end{array} \qquad \begin{array}{r} 100 \\ \times 100 \\ \hline \end{array} \qquad \begin{array}{r} 100 \\ \times 11 \\ \hline \end{array}$$

   b. The following is a multiplication table of sixes from one through five in binary notation.

110 x 1 = ?
110 x 10 = ?
110 x 11 = ?
110 x 100 = ?
110 x 101 = ?

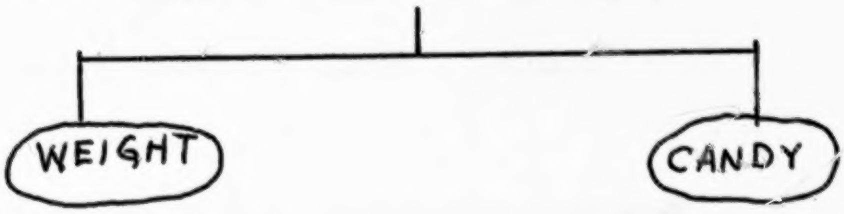
   Write the binary numerals which should follow the = sign.

## VII. Mathematics is Fun

The following recreational problems are based on the binary numbers system.

A. Problem: In an old country store candy is weighed on a balance scale. The candy is put on one pan of the scale and the weights on the other.

Find the least number of weights and their denominations.



The proprietor sells candy in amounts weighing from one ounce to fifteen ounces.

1. Discuss the weights.

   a. A 1 oz. weight would be necessary as a start. Similarly a 2 oz. weight would be necessary. Would a 3 oz. weight be necessary? (No. The 2 oz. weight and the 1 oz weight could be used to measure 3 oz.) A 4 oz. weight would be necessary since 4 could not be measured with a 1 oz. and a 2 oz. weight.

   b. Guide pupils to develop the following table of combinations of weights to fifteen.

| Number of ounces of Candy | Least No. of Weights in Ounces |
|---|---|
| 1 | 1 |
| 2 | 2 |
| 3 | 2 + 1 |
| 4 | 4 |
| 5 | 4 + 1 |
| 6 | 4 + 2 |
| 7 | 4 + 2 + 1 |
| 8 | 8 |
| 9 | 8 + 1 |
| 10 | 8 + 2 |
| 11 | 8 + 2 + 1 |
| 12 | 8 + 4 |
| 13 | 8 + 4 + 1 |
| 14 | 8 + 4 + 2 |
| 15 | 8 + 4 + 2 + 1 |

   c. Have pupil see that in the listing of weights the only numbers that appear are 1, 2, 4, 8, so that only 1 oz. weight, a 2 oz. weight, a 4 oz. weight, and an 8 oz. weight are necessary, and the minimum number of weights is four.

d. Elicit from pupil that the weights 1, 2, 4, 8 are place values in the binary system.

| Eights | Fours | Twos | Ones |
|---|---|---|---|

Have have him return to V.C. to compare the weights with the numbers in the binary system.

2. In a similar manner have pupil find the least number of weights needed to measure weights from one through thirty-one ounces. (5 weights - The additional weight is the 16 oz. weight.) Return to V.C. for comparison.

B. Reading Binary Numbers on an Electrical Hookup

Use a string of lights in a parallel circuit to represent numbers in a binary system. (A diagram on the chalkboard may also be used.)

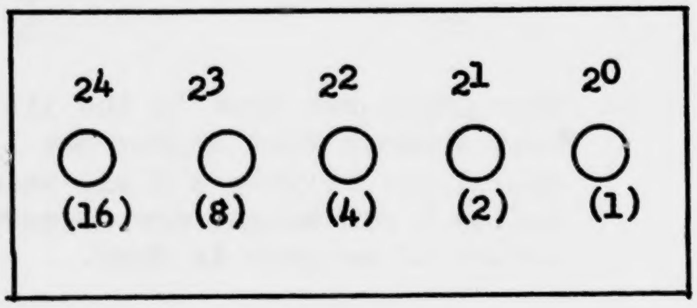
Let us agree that a lit bulb will always represent 1 in that place; and that a bulb which is unlit will represent 0 in that place.

1. Have pupil see that only two conditions are possible for each bulb in each place (lit or unlit).

2. Have him realize that the numeral system which has only two elements (numerals) in each place is the binary system.

3. Review that in the binary system the first place from right to left is ones place, the second 2's place, the third $2^2$ (4) place, the fourth place is $2^3$ (8) place and the fifth place is $2^4$ (16) place.

4. List the numbers in one-to-one correspondence with the lights.



5. Let us assume that the lights are on and off as in this illustration.

⊗ ○ ⊗ ⊗ ○

What number does this represent?

Binary System -- 1 0 1 1 0
Expansion in = (1 x 16) + (0 x 8) + (1 x 4) + (1 x 2) + (0 x 1)
Dec. System = 16 + 4 + 2
= 22

6. Practice - Change each of the following combination of lights to the number in the binary system and then find the decimal number by expanding.

| | $2^4$ (16) | $2^3$ (8) | $2^2$ (4) | $2^1$ (2) | $2^0$ (1) |
|---|---|---|---|---|---|
| a. | | * | * | | * |
| b. | | | | | * |
| c. | * | | * | | |
| d. | | * | | | |
| e. | | | * | | * |
| f. | | * | * | | * |
| g. | | | | * | * |
| h. | * | | * | | * |
| i. | | * | | | * |
| j. | | | * | * | * |
| k. | * | | | * | * |
| l. | | * | * | * | |
| m. | | | | * | |
| n. | | | * | | * |
| o. | | | * | | * |
| p. | * | | * | | * |
| q. | | * | * | * | |

7. The use of the electric tree lights should remind the pupil that in the use of all electricity, the current is either on or off. This is a choice of two conditions only. The only numeral system that has only two elements in each place is the binary system. By using as many electric lights as we please, we can represent any number no matter how large. The electronic computers use enough places to represent a number as large as 9,999,999,999. in the decimal system.

These computors are fed the numbers in the decimal system. The machines change them into the binary system and then add, subtract, multiply, or divide as instructed by the person who writes the program of instructions.

8. **Have pupil apply the arrangement of lights for decoding a message.**

a. **Have pupils list numbers and letters of the alphabet in a one-to-one correspondence. Let the number 1 represent the letter a. The following table should be set up.**

| No. | Rep. | No. | Rep. |
|---|---|---|---|
| 1 | a | 14 | n |
| 2 | b | 15 | o |
| 3 | c | 16 | p |
| 4 | d | 17 | q |
| 5 | e | 18 | r |
| 6 | f | 19 | s |
| 7 | g | 20 | t |
| 8 | h | 21 | u |
| 9 | i | 22 | v |
| 10 | j | 23 | w |
| 11 | k | 24 | x |
| 12 | l | 25 | y |
| 13 | m | 26 | z |

○ ○ ○ ○ ⊗ = a

○ ○ ○ ⊗ ○ = b

○ ○ ○ ⊗ ⊗ = c

○ ○ ⊗ ○ ○ = d

b. **Distribute a rexographed sheet of paper containing a message and help him decode it. Each line represents a letter. A double space between lines indicates the beginning of a word.**

| 16 | 8 | 4 | 2 | 1 |
|---|---|---|---|---|
| * |  | * |  |  |
|  | * |  |  |  |
|  |  | * |  | * |
|  | * | * |  | * |
|  | * | * | * | * |
| * |  |  | * |  |
|  |  | * |  | * |
| * | * |  |  | * |
|  | * | * | * | * |
| * |  | * |  | * |
|  | * | * |  |  |
|  |  | * |  | * |
|  |  |  |  | * |
| * |  |  | * |  |
|  | * | * | * |  |

(16+0+4+0+0) = 20; 20 = t

8. b.(Cont.)

| 16 | 8 | 4 | 2 | 1 |
|---|---|---|---|---|
| * |  | * |  |  |
|  | * |  |  |  |
|  |  | * |  | * |
|  | * | * |  | * |
|  | * | * | * | * |
| * |  |  | * |  |
|  |  | * |  | * |
| * | * |  |  |  |
|  | * | * | * | * |
| * |  | * |  | * |
|  |  | * |  | * |
| * |  |  | * |  |
|  | * | * | * |  |

Note: The message reads "The more you learn, the more you earn."

c. Pupils may like to code a message of their own which other pupils may decode.

Free reference for the binary number system, "coding and decoding: "Yes, No - One, Zero" -- Esso Oil Company, Hutchinson River Parkway, Pelham, New York.

C. The Peasant System of Multiplication

1. Introductory discussion

a. Have pupil consider which operation in mathematics is simplest. (Addition)

b. Have pupils multiply whole numbers by 2. They should realize that it is easy to multiply by 2.

c. Give pupils some practice in finding $\frac{1}{2}$ of a number.

d. From their experience in b and c above, pupils will agree that 2 and $\frac{1}{2}$ are easy numbers to work with.

2. **Method of multiplication by doubling**

**a. Problem:** 18 x 17

**Step 1.**

| Col I | Col. II | |
|---|---|---|
| 18 | 17 | |
| 9 | 34 | (In this step we found $\frac{1}{2}$ of 18 (9) and doubled 17 (34.) |
| 4 | 68 | (Take $\frac{1}{2}$ of 9, discarding the remainder. Double 34.) |
| 2 | 136 | (Take $\frac{1}{2}$ of 4. Double 68) |
| 1 | 272 | (Take $\frac{1}{2}$ of 2 and double 136.) |

**Step 2.**

1) Have pupil write the odd numbers in Col I

9
1

2) Have him place the corresponding numbers in Column II next to them.

| | |
|---|---|
| 9 | 34 |
| 1 | 272 |

3) Have him find the sum of the numbers on the right (306). 306 is the product of 18 and 17.

4) Have the pupil check the product by multiplying 18 by 17 in the usual manner.

$$\begin{array}{r} 17 \\ 18 \\ \hline 136 \\ 17\phantom{0} \\ \hline 306 \end{array}$$

**Problem 2:** 39 x 19

**Step 1.**

| Col.I | Col.II |
|---|---|
| 39 | 19 |
| 19 | 38 |
| 9 | 76 |
| 4 | 152 |
| 2 | 304 |
| 1 | 608 |

**Step 2.**

| | |
|---|---|
| 39 | 19 |
| 19 | 38 |
| 9 | 76 |
| 1 | 608 |
| | 741 |

Product of 39 and 19 is 741
Check by usual method.

$$\begin{array}{r} 19 \\ \underline{39} \\ 171 \\ \underline{57\phantom{0}} \\ 741 \end{array}$$

b. Develop with pupil a "proof" of this method.

1) **Problem:** 18 x 17

The numbers in Column II are

17 = 1 x 17
34 = 2 x 17
68 = 4 x 17
136= 8 x 17
272=16 x 17

Have pupil recall that he selected

$$\begin{array}{lrcr} & 34 & \text{i.e.} & 2 \times 17 \\ \text{and} & \underline{+272} & \text{i.e.} & \underline{16} \times 17 \\ & 306 & = & 18 \times 17 \end{array}$$

So that by this method he multiplied 18 by 17 as the problem called for, simplify by halving, doubling and adding.

2) Similarly for problem 2 - 39 x 19

19 = 1 x 19
38 = 2 x 19
76 = 4 x 19
152 = 8 x 19
304 = 16 x 19
608 = 32 x 19

Have pupil recall that he selected

$$\begin{array}{lrcr} & 19 & \text{i.e.} & 1 \times 19 \\ & 38 & \text{i.e.} & 2 \times 19 \\ & 76 & \text{i.e.} & 4 \times 19 \\ \text{and} & \underline{608} & \text{i.e.} & \underline{32} \times 19 \\ & 741 & = & 39 \times 19 \end{array}$$

so that by this method he multiplied 39 x 19 as the problem called for simply by halving, doubling and adding.

3. Have pupil relate this method of multiplication to binary numeral system.

   a. Have him note that in 18 x 17, above, one set of factors in the multiplication is 1, 2, 4, 8, 16, and in 39 x 19 one set of factors is 1, 2, 4, 8, 16, 32.

   b. Have him recall that these are the place values in the binary system.

   c. Have him note from these two illustrations that the sum of some of these place values is equal to a factor of the original multiplication example. For example, the factor 18 is the sum 2 + 16; the factor 39 is the sum 1 + 2 + 4 + 32.

4. Practice

   Solve by the halving and doubling method and check by decimal multiplication.

   a. 5 x 9

   b. 15 x 21

   c. 27 x 28

   d. 48 x 37